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Abstract

The aim of this work is twofold. In a first, abstract part, it is shown how to derive an asymptotic equation for the amplitude of weakly nonlinear surface waves associated with neutrally stable undercompressive shocks. The amplitude equation obtained is a nonlocal generalization of Burgers' equation, for which an explicit stability condition is exhibited. This is an extension of earlier results by J. Hunter. The second part is devoted to 'ideal' subsonic phase boundaries, which were shown by the first author to be associated with linear surface waves. The amplitude equation for corresponding weakly nonlinear surface waves is calculated explicitly and the stability condition is investigated analytically and numerically.

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1 Introduction

This work is concerned with the multi-dimensional theory of – possibly non-classical – shock waves that are neutrally stable, which means that their linearized stability analysis yields neutral normal modes. More specifically, we are interested in cases when these neutral modes are of finite energy, that is, when these modes are (genuine) *surface waves*. A program initiated by Hunter [10] has shown that surface waves are usually associated, in the weakly nonlinear regime, to *amplitude equations* that are nonlocal generalizations of Burgers' equation. Our main purpose is to apply this program

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in the framework of 'shocks', including undercompressive ones, with application to phase boundaries. Indeed, it was shown in [3] that nondissipative, dynamic and subsonic phase boundaries in van der Waals-like fluids are neutrally stable, with surface waves (also see [5]). The present paper contains two main parts. In the first one we derive the amplitude equation associated with surface waves along neutrally stable shocks in an abstract framework, and give an alternative version of Hunter's stability condition that is easy to check in practice. In the second part we perform the computations in the explicit case of surface waves along dynamic subsonic phase boundaries. It turns out, as our numerical results show, that Hunter's stability condition is not satisfied by the amplitude equation associated with subsonic phase boundaries. This is in contrast with what happens in Elasticity for instance, where the amplitude equation associated with Rayleigh waves is known to satisfy Hunter's condition [10, 14, 15].

2 Derivation of the amplitude equation

2.1 General framework

We consider a hyperbolic system of conservation laws

$$\sum_{i=0}^{d} \partial_i f^i(u) = 0_n \quad , \qquad x \in \mathbb{R}^d \, , \tag{2.1}$$

where the unknown is $u = {}^t(u_1, \dots, u_n) \colon (t,x) \in [0,\infty) \times \mathbb{R}^d \mapsto u(t,x) \in \mathbb{R}^n$, ∂_0 stands for the partial derivative with respect to t and ∂_i denotes the partial derivative with respect to x_i, $i = 1, \dots, d$. Here, $f^i = {}^t(f^i_1, \dots, f^i_n) \colon u \in \mathcal{U} \mapsto f^i(u) \in \mathbb{R}^n$, $i = 0, \dots, d$, are given smooth fluxes (at least $\mathscr{C}^2$) on an open subset $\mathcal{U}$ of $\mathbb{R}^n$. We shall denote by $A^i := (\partial_{u_k} f^i_j)_{1 \le j,k \le n} \colon u \in \mathcal{U} \mapsto A^i(u) \in \mathbb{R}^{n \times n}$ the Jacobian matrix of f^i, $i = 0, \dots, d$, and assume that:

- for all $u \in \mathcal{U}$, the matrix $A^0(u)$ is nonsingular,
- for all $u \in \mathcal{U}$ and all $\eta \in \mathbb{R}^d \setminus \{0_d\}$, the matrix $A^0(u)^{-1} \sum_{i=1}^d \eta_i A^i(u)$, has n real eigenvalues $\lambda_1(u,\eta) \le \lambda_2(u,\eta) \le \ldots \le \lambda_n(u,\eta)$ and n linearly independent corresponding eigenvectors $r_1(u,\eta), \dots, r_n(u,\eta) \in \mathbb{R}^n$, i.e.

$$\left(\sum_{i=1}^{d} \eta_i A^i(u) - \lambda_j A^0(u) \right) r_j = 0_n \quad , \quad j = 1, \dots, n \, .$$

We are concerned here with special, shock-like weak solutions to (2.1) that are $\mathscr{C}^1$ outside a smooth moving interface. Recall that for a hypersurface

$$\Sigma := \Big\{ (t,x) \in [0,\infty) \times \mathbb{R}^d \, : \, \Phi(t,x) = 0 \Big\} \, , \tag{2.2}$$

where $\Phi\colon [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ is a $\mathscr{C}^1$ function, a mapping $u:(0,T)\times\mathbb{R}^d \to \mathbb{R}^n$ that is $\mathscr{C}^1$ on either side of Σ is a weak solution of (2.1), if and only if,

$$\sum_{i=0}^{d} \partial_i f^i(u_\pm) = 0_n \quad , \quad \pm\Phi(t,x) > 0 \ , \ t \in (0,T) \ ,$$

where $u_\pm$ is the restriction of u to the domain

$$\Omega_\pm := \Big\{(t,x) \in [0,\infty)\times\mathbb{R}^d \ : \ \pm\,\Phi(t,x) > 0\Big\} \ ,$$

and

$$\sum_{i=0}^{d} \big[f^i(u)\big]\,\partial_i\Phi = 0_n \quad , \quad \Phi(t,x) = 0 \ , \tag{2.3}$$

where the brackets $[\,\cdot\,]$ give the 'strength' of the jump across the interface.

It is well-known that the Rankine-Hugoniot jump conditions in (2.3) are not sufficient in general to ensure uniqueness of weak solutions. They must be supplemented with admissibility conditions. For "classical" shocks, standard admissibility conditions are given by the Lax inequalities (see for instance [16]), which require that the number of characteristics outgoing the shockfront is less than the number of incoming characteristics. More precisely, for Laxian shocks in a states space of dimension n, the number of outgoing characteristics is $n-1$ and the number of incoming ones is $n+1$. For "nonclassical" shocks, the situation is different, and in particular for *undercompressive* ones, the number in of incoming characteristics is less or equal to n. Then a number $p := n+1-in$ of additional jump conditions is needed. In what follows, we consider undercompressive shocks for which these additional jump conditions can be written as

$$\sum_{i=0}^{d} \big[g^i(u)\big]\,\partial_i\Phi = 0_p \quad , \quad \Phi(t,x) = 0 \ , \tag{2.4}$$

where $g^i\colon u \in \mathcal{U} \mapsto g^i(u) \in \mathbb{R}^p$, $i = 0,\ldots,d$, are smooth (at least $\mathscr{C}^2$). For both Laxian and undercompressive shocks, the resulting system is

$$\left\{\begin{array}{lcll} \displaystyle\sum_{i=0}^{d} \partial_i f^i(u) & = & 0_n & , \quad \Phi(t,x) \neq 0\,, \\ \displaystyle\sum_{i=0}^{d} \Big[\widetilde{f}^i(u)\Big]\,\partial_i\Phi & = & 0_{n+p} & , \quad \Phi(t,x) = 0\,, \end{array}\right. \tag{2.5}$$

where

- for classical shocks: $p = 0$, $\widetilde{f}^i(u) := f^i(u) \in \mathbb{R}^n$;

- for undercompressive shocks: $p \geq 1$, $\widetilde{f}^i(u) := \begin{pmatrix} f^i(u) \\ g^i(u) \end{pmatrix} \in \mathbb{R}^{n+p}$.

This work is motivated by nondissipative subsonic phase boundaries in van der Waals fluids, which can viewed as undercompressive shocks with $p = 1$ and (2.4) given by the so-called capillarity criterion [3]. More precisely, for isothermal phase boundaries, the interior equations are given by the conservation of mass and of momentum – with a non-monotone pressure law $\rho \mapsto p(\rho)$ – and the additional jump condition is given by the conservation of total energy (in fact, the free energy plus the kinetic energy).

Our purpose is to describe nontrivial approximate solutions to the fully nonlinear problem (2.5). The starting point will be a planar stationary noncharacteristic shock-like solution.

Assumption 1 *There exists $\underline{u} = (u_l, u_r) \in \mathbb{R}^n \times \mathbb{R}^n$ such that*

$$\begin{cases} u(t,x) := \begin{cases} u_l & , \quad x_d < 0 \, , \\ u_r & , \quad x_d > 0 \, , \end{cases} \\ \Phi(t,x) := x_d \, , \end{cases}$$

is a solution of the nonlinear problem (2.5). *In addition, we assume that the matrices $A^d(u_l)$ and $A^d(u_r)$ are nonsingular.*

2.2 The linearized problem

We are interested in solutions of (2.5) close to the planar stationary solution $\underline{u}$ given by Assumption 1. In this respect, we shall concentrate on solutions (v, Ψ) for which the location of the shock front is given by an equation of the form

$$\Psi(t,x) \, = \, 0 \, , \quad \text{where} \;\; \Psi(t,x) \, = \, x_d \, - \, \chi(t, x_1, \ldots, x_{d-1})$$

for a smooth map $\chi : (t, x_1, \ldots, x_{d-1}) \in [0,\infty) \times \mathbb{R}^{d-1} \mapsto \chi(t, x_1, \ldots, x_{d-1}) \in \mathbb{R}$. Then the system (2.5) applied to (v, Ψ) instead of (u, Φ) becomes

$$\begin{cases} \displaystyle\sum_{i=0}^{d} \partial_i f^i(v) \quad = \quad 0_n \qquad , \quad x_d \neq \chi(t, x_1, \ldots, x_{d-1}) \, , \\ \displaystyle\sum_{i=0}^{d-1} \left[\widetilde{f}^i(v)\right] \partial_i \chi \quad = \quad \left[\widetilde{f}^d(v)\right] \quad , \quad x_d = \chi(t, x_1, \ldots, x_{d-1}) \, , \end{cases} \tag{2.6}$$

where $[\widetilde{f}^i(v)] := \widetilde{f}^i(v_r) - \widetilde{f}^i(v_l) \in \mathbb{R}^{n+p}$, being

$$v_l(t, x_1, \ldots, x_{d-1}) := \lim_{x_d \nearrow \chi(t, x_1, \ldots, x_{d-1})} v(t, x_1, \ldots, x_d) \, ,$$
$$v_r(t, x_1, \ldots, x_{d-1}) := \lim_{x_d \searrow \chi(t, x_1, \ldots, x_{d-1})} v(t, x_1, \ldots, x_d) \, .$$

As usual for free boundary value problems, we start by making a change of variables that leads to a problem in a *fixed* domain. Introducing the new unknowns $v_\pm\colon [0,\infty)\times\mathbb{R}^{d-1}\times[0,\infty)\to\mathbb{R}^n$, related to v by

$$v_\pm(y_0,y_1,\dots,y_d) := v(y_0,\dots,y_{d-1},\chi(y_0,\dots,y_{d-1})\pm y_d)\,,$$

and redefining v as

$$v := (v_-,v_+)\colon [0,\infty)\times\mathbb{R}^{d-1}\times[0,\infty)\to\mathbb{R}^{2n}\,,$$

we are led to the boundary value problem

$$\left\{\begin{array}{lcl} \mathbb{L}(v,\nabla\chi)\cdot v &=& 0_{2n} \quad,\quad y_d>0\,,\\ b(v,\nabla\chi) &=& 0_{n+p} \quad,\quad y_d=0\,, \end{array}\right. \tag{2.7}$$

with

$$\begin{array}{l} \mathbb{L}(v,\nabla\chi) := \displaystyle\sum_{i=0}^{d-1}\mathbb{A}^i(v)\partial_{y_i}+\mathbb{A}^d(v,\nabla\chi)\partial_{y_d}\\ b(v,\nabla\chi) := \displaystyle\sum_{i=0}^{d-1}(\partial_i\chi)\left[\widetilde{f}^i(v)\right]-\left[\widetilde{f}^d(v)\right] \qquad\in\quad \mathbb{R}^{n+p}\,, \end{array}$$

where, for $i=0,\dots,d$,

$$\mathbb{A}^i(v) := \left(\begin{array}{c|c} A^i(v_-) & 0_{n\times n}\\ \hline 0_{n\times n} & A^i(v_+)\end{array}\right)\quad,\quad \breve{\mathbb{I}}_{2n} := \left(\begin{array}{c|c} -\mathrm{I}_n & 0_{n\times n}\\ \hline 0_{n\times n} & \mathrm{I}_n\end{array}\right)$$

$$\breve{\mathbb{A}}^i(\underline{u}) := \breve{\mathbb{I}}_{2n}\mathbb{A}^i(\underline{u})\,,\ \mathbb{A}^d(v,\nabla\chi) := \breve{\mathbb{A}}^d(v)-\sum_{i=0}^{d-1}(\partial_i\chi)\breve{\mathbb{A}}^i(v)\in\mathbb{R}^{2n\times 2n}.$$

Using an observation of Métivier [13], we may simplify the boundary conditions in (2.7), at least for solutions close to $\underline{u}$, provided that the following assumption holds true.

Assumption 2 *The jump vectors* $[\widetilde{f}^0(\underline{u})],\dots,[\widetilde{f}^{d-1}(\underline{u})]$ *are independent in* $\mathbb{R}^{n+p}$.

Under Assumption 2, there exist a neighborhood $\mathcal{V}\subseteq\mathcal{U}\times\mathcal{U}$ of $\underline{u}$ and a map $Q: v\in\mathcal{V}\mapsto Q(v)\in\mathrm{GL}_{n+p}(\mathbb{R})$ (the group of non-singular $(n+p)\times(n+p)$ matrices) such that

$$Q(v)\sum_{i=0}^{d-1}\xi_{i+1}\left[\widetilde{f}^i(v)\right] = \begin{pmatrix}\xi\\ 0_{n+p-d}\end{pmatrix}\in\mathbb{R}^{n+p}\qquad\text{for all }\xi\in\mathbb{R}^d\text{ and all }v\in\mathcal{V}.$$

Therefore, the boundary value problem (2.7) can be rewritten with “simpler” boundary conditions:

$$\left\{\begin{array}{l} \mathbb{L}(v,\nabla\chi)\cdot v = 0_{2n} \quad,\quad y_d>0\,,\\ J\nabla\chi+h(v) = 0_{n+p} \quad,\quad y_d=0\,, \end{array}\right. \tag{2.8}$$

where

$$J := \left(\frac{\mathrm{I}_d}{0_{(n+p-d)\times d}} \right) \in \mathbb{R}^{(n+p)\times d} \quad , \quad h(v) := -Q(v) \left[\widetilde{f}^d(v) \right] \in \mathbb{R}^{n+p} \, .$$

The linearization of the simplified problem (2.8) about its constant solution ($v \equiv \underline{u}, \chi \equiv 0$) readily gives the equations for the perturbations $\dot{v}$ and $\dot{\chi}$ of v and χ respectively,

$$\left\{ \begin{array}{lll} \mathbb{L}(\underline{u}, 0) \cdot \dot{v} = 0_{2n} & , & y_d > 0 \, , \\ J \nabla \dot{\chi} + H(\underline{u}) \cdot \dot{v} = 0_{n+p} & , & y_d = 0 \, , \end{array} \right. \tag{2.9}$$

where $H(u) \in \mathbb{R}^{(n+p)\times 2n}$ denotes the Jacobian matrix of h at u.

We shall now make a further assumption regarding the solutions of (2.9) that go to zero as y_d goes to $+\infty$. First of all, we introduce, for $\eta = (\eta_0, \eta_1, \dots, \eta_{d-1}) \in \mathbb{R} \times \mathbb{R}^{d-1}$, the operator

$$\widehat{\mathbb{L}}(\underline{u}, \eta) \; := \; \mathbb{A}(u, \mathrm{i}\eta) \; + \; \breve{\mathbb{A}}^d(\underline{u}) \, \partial_{y_d} \, , \quad \text{with } \; \mathbb{A}(\underline{u}, \mathrm{i}\eta) \; := \sum_{k=0}^{d-1} \mathrm{i} \, \eta_k \; \mathbb{A}^k(\underline{u}) \, ,$$

obtained from $\mathbb{L}(\underline{u}, 0)$ by Fourier transform in the tangential variable $y = (y_0, y_1, \dots, y_{d-1})$. Observe that by the noncharacteristicity of the shock $\underline{u}$ (Assumption 1), the $(2n \times 2n)$ block-diagonal matrix $\breve{\mathbb{A}}^d(\underline{u})$ is nonsingular. In what follows, we also use the notation $\widehat{\mathbb{L}}(\underline{u}, \eta)$ for vectors η for which $\eta_0 = -i\tau \in \mathbb{C}$, $\mathrm{Re}(\tau) > 0$, the operator $\widehat{\mathbb{L}}(\underline{u}, \tau, \mathrm{i}\eta_1, \cdots, \mathrm{i}\eta_{d-1})$ arising when we perform a Laplace transform in y_0 instead of a Fourier transform. The hyperbolicity of (2.1) implies, by a classical observation due to Hersh [9], that for all $\eta = (\eta_0, \eta_1, \dots, \eta_{d-1}) \in \mathbb{C} \times \mathbb{R}^{d-1}$ with $\mathrm{Im}(\eta_0) \; < \; 0$, the matrix

$$\mathcal{A}(\underline{u}, \eta) \; := \; - \, \breve{\mathbb{A}}^d(\underline{u})^{-1} \, \mathbb{A}(\underline{u}, \mathrm{i}\eta) \tag{2.10}$$

is hyperbolic, that is, has no purely imaginary spectrum. It is well known that the well-posedness of the linear problem (2.9) crucially depends on the properties of the invariant subspaces of $\mathcal{A}(\underline{u}, \eta)$. The following is a natural generalization of the Lopatinskiĭ condition to undercompressive shocks [8] (regarding Laxian shocks, see the seminal work by [12]).

Assumption 3 *For all $\eta = (\eta_0, \eta_1, \dots, \eta_{d-1}) \in \mathbb{C} \times \mathbb{R}^{d-1}$ with $\mathrm{Im}(\eta_0) \; < \; 0$, the stable subspace $\mathcal{E}_s(\underline{u}, \eta)$ of $\mathcal{A}(\underline{u}, \eta)$ is of dimension $q := n + p - 1$,* and *there is no nontrivial $(X, V) \in \mathbb{C} \times \mathcal{E}_s(\underline{u}, \eta)$ such that*

$$X J \eta + H(\underline{u}) V = 0_{n+p} \, . \tag{2.11}$$

Assumption 3 is known to be necessary for the well-posedness of (2.9) associated with suitable initial data. To investigate the actual well-posedness of this initial-boundary-value problem we need to go further and consider the subspace $\mathcal{E}(\underline{u}, \eta)$ obtained as

$$\mathcal{E}(\underline{u}, \eta) := \lim_{b \nearrow 0} \mathcal{E}_s(\underline{u}, \eta_0 + ib, \eta_1, \ldots, \eta_{d-1})$$

(in the Grassmannian of q-dimensional subspaces of $\mathbb{C}^{2n}$). As the hyperbolicity of the matrix $\mathcal{A}(\underline{u}, \eta)$ fails in general for real η_0, the limiting space $\mathcal{E}(\underline{u}, \eta)$ decomposes as

$$\mathcal{E}(\underline{u}, \eta) = \mathcal{E}_-(\underline{u}, \eta) \oplus \mathcal{E}_0(\underline{u}, \eta),$$

where $\mathcal{E}_-(\underline{u}, \eta)$ is the (genuine) stable subspace of $\mathcal{A}(\underline{u}, \eta)$, of dimension say $m \leq q$, and $\mathcal{E}_0(\underline{u}, \eta)$ is a subspace of the center subspace of $\mathcal{A}(\underline{u}, \eta)$.

Assumption 4 *There exists $\eta = (\eta_0, \eta_1, \ldots, \eta_{d-1}) \in \mathbb{R}^d$, $\eta_0 \neq 0$, and $(X_\eta, V_\eta) \in \mathbb{C} \times \mathcal{E}(\underline{u}, \eta)$ such that*

$$\{ (X, V) \in \mathbb{C} \times \mathcal{E}(\underline{u}, \eta) ; \; XJ\eta + H(\underline{u})V = 0 \} = \mathbb{C} \{(X_\eta, V_\eta)\},$$

and *the vector V_η belongs to $\mathcal{E}_-(\underline{u}, \eta) \backslash \{0_{2n}\}$.*

Assumption 4 means that (2.5) admits surface waves, that is, solutions that are exponentially decaying in y_d and oscillating in $y = (y_0, y_1, \ldots, y_{d-1})$. As observed in [6, Chap. 7], even though surface waves signal a failure of the so-called *uniform* Kreiss-Lopatinskiĭ condition, their existence is still compatible with the well-posedness of constant-coefficients linear *homogeneous* boundary value problems, such as (2.9). For non-linear problems, the resolution of which relies on non-homogeneous linear problems, surface waves are responsible for a loss of regularity, see in particular the work of Coulombel and Secchi [7].

Our purpose here is to adapt the method proposed by Hunter [10] to derive an amplitude equation for weakly non-linear surface waves associated with weakly stable shocks – i.e. shocks satisfying in particular Assumption 4.

Finally, we shall assume that frequencies of surface waves do not correspond to 'glancing points'. This is the purpose of the following.

Assumption 5 *For all η as in Assumption 4, the matrix $\mathcal{A}(\underline{u}, \eta)$ is diagonalizable.*

In particular, for nondissipative isothermal subsonic phase transitions considered, our five assumptions are satisfied; see Section 3 for more details. The existence of surface waves has also been evidenced by Serre [17] in a general framework, when the evolution equations derive from a variational principle.

We enter now into more technical details. Assumption 5 and the fact that $\mathcal{A}(\underline{u}, \eta)$ has purely imaginary coefficients implies the existence of eigenvalues

$\beta_i^\pm \in \mathbb{C}$ and associated eigenvectors $R_i^\pm \in \mathbb{C}^{2n}$, for $i \in \{1, \ldots, q_\pm\}$ with $q_- := q = n + p - 1$, $q_+ := n - p + 1$,

$$(\mathcal{A}(\underline{u}, \eta) \; - \; \beta_i^\pm \, \mathrm{I}_{2n} \,) \, R_i^\pm \; = \; 0_{2n} \, ,$$

or equivalently,

$$(\, \mathbb{A}(\underline{u}, \mathrm{i}\eta) \; + \; \beta_i^\pm \, \check{\mathbb{A}}^d(\underline{u}) \,) \, R_i^\pm \; = \; 0_{2n} \, ,$$

with

$$\mathrm{Re}(\beta_i^\pm) \gtrless 0 \, , \quad \beta_i^+ = -\overline{\beta_i^-} \, , \quad R_i^+ = \overline{R_i^-} \, , \quad i \in \{1, \ldots, m\} \, ,$$

$$\mathrm{Re}(\beta_i^\pm) = 0 \, , \quad R_i^\pm \in \mathbb{R}^{2n} \, , \quad i \in \{m+1, \, \ldots, q_\pm\} \, ,$$

and

$$\mathbb{C}^{2n} \; = \; \mathrm{Span}\{R_1^-, \ldots, R_{q_-}^-, R_1^+, \ldots, R_{q_+}^+\} \; = \; \mathcal{E}_-(\underline{u}, \eta) \, \oplus \mathcal{E}_c(\underline{u}, \eta) \, \oplus \, \mathcal{E}_+(\underline{u}, \eta) \, ,$$

where

$$\mathcal{E}_\pm(\underline{u}, \eta) \; := \; \mathrm{Span}\{R_1^\pm, \ldots, R_m^\pm\} \, ,$$

(we recall that $\mathcal{E}_-(\underline{u}, \eta)$ is the stable subspace of $\mathcal{A}(\underline{u}, \eta)$, and similarly, $\mathcal{E}_+(\underline{u}, \eta)$ is its unstable subspace), and

$$\mathcal{E}_c(\underline{u}, \eta) \; := \; \mathrm{Span}\{R_{m+1}^-, \ldots, R_{q_-}^-, R_{m+1}^+, \ldots, R_{q_+}^+\}$$

is the center subspace of $\mathcal{A}(\underline{u}, \eta)$.

Let $L_i^\pm \in \mathbb{C}^{2n}$ be such that $L_i^\pm \, \check{\mathbb{A}}^d(\underline{u})$ are left eigenvectors of the matrix $\mathcal{A}(\underline{u}, \eta)^*$, and more precisely,

$$(L_i^\pm)^* \, (\, \mathbb{A}(\underline{u}, \mathrm{i}\eta) \; + \; \beta_i^\pm \, \check{\mathbb{A}}^d(\underline{u}) \,) \; = \; 0_{2n}^* \, .$$

Above "$*$" gives the conjugate of the transpose, i.e. $A^* \; = \; {}^t(\overline{A})$. Like the right eigenvectors, they can be chosen so that $L_i^+ = \overline{L_i^-}$, $i = 1, \ldots, m$, and $L_i^\pm \in \mathbb{R}^{2n}$, $i = m+1, \ldots, q_\pm$. We make the following further assumption.

Assumption 6

$$(L_i^\pm)^* \, \check{\mathbb{A}}^d(\underline{u}) \, R_j^\pm = 0 \, , \quad i, j = 1, \ldots, q_\pm \, , \; i \neq j \, ,$$

$$(L_i^\pm)^* \, \check{\mathbb{A}}^d(\underline{u}) \, R_j^\mp = 0 \, , \, i = 1, \ldots, q_\pm, \; j = 1, \ldots, q_\mp \, .$$

Observe that Assumption 6 is automatic if all the $\beta_i^\pm$ are distinct. We rescale the eigenvectors so that

$$(L_j^\pm)^* \, \check{\mathbb{A}}^d(\underline{u}) \, R_j^\pm = 1 \, , \quad j = 1, \ldots, q_\pm \, .$$

Now, Assumption 4 may be interpreted in terms of the eigenvectors $R_1^-, \ldots, R_q^-$ only. We first make some further reductions. Observing that

$\mathcal{A}(\underline{u}; k\eta) = k\,\mathcal{A}(\underline{u}, \eta)$ for any $k \in \mathbb{R}$ (which is due to scale invariance), we see that the subspace $\mathcal{E}(\underline{u}, \eta)$ is positively homogeneous degree 0 in η. Therefore, the wave vectors η for which Assumption 4 holds true form a positive cone, and for all $k > 0$

$$X_{k\eta} = \frac{1}{k} X_\eta , \quad V_{k\eta} = V_\eta .$$

Thus, without loss of generality, we may assume that $\eta_0 = 1$. Then we observe that (2.11) equivalently reads

$$X = -H^1(\underline{u})\,V , \qquad \mathcal{C}(\underline{u}, \eta)\,V = 0 , \tag{2.12}$$

where $H^1(\underline{u}) = \mathrm{d}h^1(\underline{u})$ is the first row of the Jacobian matrix $H(\underline{u}) = \mathrm{d}h(\underline{u})$, and $\mathcal{C}(\underline{u}, \eta) \in \mathbb{R}^{q\times 2n}$ is defined by

$$\mathcal{C}(\underline{u}, \eta) := T(\eta)\,H(\underline{u}) , \quad T(\eta) := \left(\begin{array}{c|c} -\eta_1 & \\ \vdots & \mathrm{I}_q \\ -\eta_{d-1} & \\ 0_{q-d+1} & \end{array}\right) \in \mathbb{R}^{q\times(q+1)} .$$

Hence, by Assumption 4, there exists $\gamma \in \mathbb{C}^m\backslash\{0_m\}$ such that

$$\sum_{j=1}^{m} \gamma_j\, \mathcal{C}(\underline{u}, \eta) R_j^- = 0_q ,$$

the components γ_j of γ merely being the components of V_η in the basis $\{R_1^-, \ldots, R_m^-\}$ of $\mathcal{E}_-(\underline{u}, \eta)$. Moreover, Assumption 4 means that the $q\times q$ matrix $[\mathcal{C}(\underline{u}, \eta)R_1^-, \ldots, \mathcal{C}(\underline{u}, \eta)R_q^-]$ is of rank $q-1$, so that there exists $\sigma \in \mathbb{C}^q\backslash\{0_q\}$ such that

$$\sigma^*\mathcal{C}(\underline{u}, \eta)R_j^- = 0 , \; j \in \{1, \ldots, q\} . \tag{2.13}$$

Since the matrix $\mathcal{C}(\underline{u}, \eta)$ and the vectors $R_{m+1}^-, \ldots, R_q^-$ have real coefficients, and $R_j^- = \overline{R_j^+}$ for $j \in \{1, \ldots, m\}$, we also have, by conjugation,

$$\begin{aligned} \overline{\sigma}^*\mathcal{C}(\underline{u}, \eta)R_j^+ &= 0 , \; j \in \{1, \ldots, m\} , \\ \overline{\sigma}^*\mathcal{C}(\underline{u}, \eta)R_k^- &= 0 , \; k \in \{m+1, \ldots q\} . \end{aligned} \tag{2.14}$$

2.3 Weakly nonlinear surface waves

We can now turn to the derivation of an amplitude equation for weakly nonlinear surface waves in (2.5). Following Hunter's approach [10], we consider an expansion for v, χ, of the form

$$v^\varepsilon(y) = \underline{u} + \varepsilon v_1(\eta_0 y_0 + \check{\eta}\cdot\check{y}, y_d, \varepsilon y_0) + \varepsilon^2 v_2(\eta_0 y_0 + \check{\eta}\cdot\check{y}, y_d, \varepsilon y_0) + \mathcal{O}(\varepsilon^3) ,$$

$$\chi^\varepsilon(y) = \varepsilon \chi_1(\eta_0 y_0 + \check{\eta}\cdot\check{y}, \varepsilon y_0) + \varepsilon^2 \chi_2(\eta_0 y_0 + \check{\eta}\cdot\check{y}, \varepsilon y_0) + \mathcal{O}(\varepsilon^3) ,$$

where $\check{\eta}$ and $\check{y}$ stand for the $(d-1)$-dimensional vectors defined by $\check{\eta} = (\eta_1, \ldots, \eta_{d-1})$ and $y = (y_1, \ldots, y_{d-1})$, and $v_{1,2}$ is supposed to go to zero as y_d goes to infinity. The above ansatz for v describes a small amplitude wave that is changing slowly in reference frame moving with the wave.

From now on, we use the notation $(\xi, z, \tau) = (\eta_0 \cdot y_0 + \check{\eta} \cdot \check{y}, y_d, \varepsilon y_0)$ for the new independent variables. Using Taylor expansions for f^i and h,

$$\mathbb{A}^i(v^\varepsilon) = \mathbb{A}^i(\underline{u}) + \varepsilon \, \mathrm{d}\mathbb{A}^i(\underline{u}) \cdot v_1 + \mathcal{O}(\varepsilon^2) \,,$$

$$h(v^\varepsilon) \; = \varepsilon H(\underline{u}) \cdot v_1 \; + \; \varepsilon^2 \; (H(\underline{u}) \cdot v_2 + \tfrac{1}{2} \, \mathrm{d}^2 h(\underline{u}) \cdot (v_1, v_1)) + \mathcal{O}(\varepsilon^3) \,,$$

and equating to zero the coefficients of ε and ε^2 in (2.8), we find

$$\left\{ \begin{array}{lcll} \mathcal{L}(\underline{u}, \eta) v_1 & = & 0_{2n} & , \quad z > 0 \\ J\eta \, \partial_\xi \chi_1 + H(\underline{u}) v_1 & = & 0_{n+p} & , \quad z = 0 \,, \end{array} \right. \tag{2.15}$$

and

$$\left\{ \begin{array}{lcll} \mathcal{L}(\underline{u}, \eta) \cdot v_2 + \mathcal{M}(\underline{u}, \eta; v_1, \partial_\xi \chi_1) \cdot v_1 & = & 0_{2n} & , \quad z > 0 \\ J\eta \, \partial_\xi \chi_2 + H(\underline{u}) \cdot v_2 + G(\underline{u}; v_1, \partial_\tau \chi_1) & = & 0_{n+p} & , \quad z = 0 \,, \end{array} \right. \tag{2.16}$$

where

$$\mathcal{L}(\underline{u}, \eta) := \mathbb{A}(\underline{u}, \eta) \, \partial_\xi + \breve{\mathbb{A}}^d(\underline{u}) \, \partial_z \,,$$

$$\begin{aligned} \mathcal{M}(\underline{u}, \eta; v_1, \partial_\xi \chi_1) := \; & \mathbb{A}^0(\underline{u}) \partial_\tau \; + \; \mathrm{d}\mathbb{A}(\underline{u}, \eta) \cdot v_1 \cdot \partial_\xi \; + \; \mathrm{d}\breve{\mathbb{A}}^d(\underline{u}) \cdot v_1 \cdot \partial_z \\ & - (\partial_\xi \chi_1) \, \breve{\mathbb{A}}(\underline{u}, \eta) \, \partial_z \,, \quad \text{with } \breve{\mathbb{A}}(\underline{u}, \eta) \; := \; \mathbb{I}_{2n} \mathbb{A}(\underline{u}, \eta) \,, \end{aligned}$$

and

$$G(\underline{u}; v_1, \partial_\tau \chi_1) := (\partial_\tau \chi_1) \, \mathbf{e}_1 + \frac{1}{2} \, \mathrm{d}^2 h(\underline{u}) \cdot (v_1, v_1) \,,$$

with $\mathbf{e}_1$ denoting the first vector of the canonical basis in $\mathbb{C}^{q+1}$.

We recall that by definition,

$$V_\eta \; = \; \sum_{j=1}^m \gamma_j \, R_j^-$$

and $X_\eta = -H^1(\underline{u}) \, V_\eta$ solve (2.11), or equivalently, (2.12). Denoting by P_j and Q_j the real and imaginary parts, respectively, of $\gamma_j \, R_j^-$, we have

$$V_\eta \; = \; P + \mathrm{i} \, Q \,, \quad \text{with } \; P := \sum_{j=1}^m P_j \,, \quad Q \; := \; \sum_{j=1}^m Q_j \,.$$

For convenience, we also introduce the notations ϱ_j and δ_j for the real and imaginary parts, respectively, of β_j^- $(j \in \{1, \ldots, m\})$. In what follows, $\mathcal{H}$ stands for the Hilbert transform, such that for any L^2 function w,

$$\mathcal{F}\mathcal{H}[w](k) \; = \; -\mathrm{i} \; \mathrm{sgn}(k) \; \mathcal{F}[w](k) \,, \; \forall k \in \mathbb{R} \,,$$

where $\mathcal{F}$ denotes the Fourier transform, with the convention

$$\mathcal{F}[w](k) \,=\, \widehat{w}(k) := \int_{-\infty}^{+\infty} w(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\,,\ \forall k \in \mathbb{R}\,.$$

Proposition 2.1 *The solutions $(\xi, z) \mapsto (v_1, \chi_1)(\xi, z)$ of (2.15) that are square integrable in ξ and such that v_1 goes to zero as $z \to +\infty$ are of the form*

$$v_1(\xi, z) \,=\, (w *_\xi r)(\xi, z)\,, \quad r(\xi, z) \,:=\, -\frac{1}{\pi}\sum_{j=1}^{m} \frac{z\,\varrho_j\,P_j \,+\, (z\,\delta_j + \xi)\,Q_j}{(\varrho_j z)^2 + (\delta_j z + \xi)^2}$$

or equivalently,

$$\widehat{v_1}(k, z) \,=\, \widehat{w}(k)\,\widehat{r}(k, z)\,,\ \widehat{r}(k, z) \,=\, \begin{cases} \displaystyle\sum_{j=1}^{m} \gamma_j\,\mathrm{e}^{\beta_j^-\,k\,z}\,R_j^-\,, & k \,>\, 0\,, z > 0\,,\\ \displaystyle\sum_{j=1}^{m} \overline{\gamma_j}\,\mathrm{e}^{\beta_j^+\,k\,z}\,R_j^+\,, & k \,<\, 0\,, z > 0\,, \end{cases}$$

and

$$v_1(\xi, 0) \,=\, w(\xi)\,P \,-\, \mathcal{H}[w](\xi)\,Q\,, \quad \chi_1(\xi) \,=\, H^1(\underline{u}) \int_{\xi}^{+\infty} v_1(\zeta, 0)\,\mathrm{d}\zeta\,,$$

where w is an arbitrary L^2 function.

Proof. By Fourier transform in the variable ξ, (2.15) becomes

$$\begin{cases} k\,\mathbb{A}(\underline{u}, \mathrm{i}\eta)\,\widehat{v_1}(k, z) \,+\, \breve{\mathbb{A}}^d(\underline{u})\,\partial_z\widehat{v_1}(k, z) \,=\, 0_{2n}\,, & z > 0\,,\\ \mathrm{i}\,k\,\widehat{\chi_1}(k)\,J\eta + H(\underline{u})\widehat{v_1}(k, 0) \,=\, 0_{n+p}\,. \end{cases} \tag{2.17}$$

Similarly as in (2.12), we may eliminate $\widehat{\chi_1}$ from the boundary condition in (2.17). We thus obtain

$$\widehat{\chi_1}(k) \,=\, \frac{\mathrm{i}}{k}\,H^1(\underline{u})\,\widehat{v_1}(k, 0)\,, \quad \mathcal{C}(\underline{u}, \eta)\,\widehat{v_1}(k, 0) \,=\, 0\,. \tag{2.18}$$

Since by Assumption 1 the matrix $\breve{\mathbb{A}}^d(\underline{u})$ is nonsingular, the first line in (2.17) is a genuine ODE on $\widehat{v_1}$, which may equivalently be written as

$$\partial_z\widehat{v_1} = \mathcal{A}(\underline{u}, k\eta)\,\widehat{v_1}\,, \tag{2.19}$$

where $\mathcal{A}(\underline{u}, k\eta)$ is defined as in (2.10) (note that $\mathcal{A}$ is homogeneous degree one in η). Then, the vanishing of $\widehat{v_1}$ at $z = +\infty$ implies that for $k > 0$, there exists $W(k) \in \mathbb{C}$ such that

$$\widehat{v_1}(k, 0) \,=\, W(k)\,V_\eta \,=\, W(k)\,\sum_{j=1}^{m} \gamma_j\,R_j^-\,,$$

hence, by the solving (2.19),

$$\widehat{v_1}(k,z) \;=\; \exp(z\,\mathcal{A}(\underline{u};k\eta))\,\widehat{v_1}(k,0) \;=\; W(k)\,\sum_{j=1}^{m}\gamma_j\,\mathrm{e}^{\beta_j^-\,k\,z}\,R_j^-\,.$$

Observing that $(k,z)\mapsto\overline{\widehat{v_1}(-k,z)}$ solves the same problem as $(k,z)\mapsto\widehat{v_1}(k,z)$, we find that for $k<0$, there exists also $W(k)\in\mathbb{C}$ such that

$$\widehat{v_1}(k,z) \;=\; W(k)\,\sum_{j=1}^{m}\overline{\gamma_j}\,\mathrm{e}^{\beta_j^+\,k\,z}\,R_j^+\,,\quad z\geq 0\,.$$

The conclusion follows by inverse Fourier transform, with $w := \mathcal{F}^{-1}[W]$. Details are standard and left to the reader. □

Proposition 2.2 *We assume that* (v_1,χ_1) *is a family of solutions of* (2.15) *as in Proposition 2.1, depending smoothly on the parameter* τ, *and that* (2.16) *admits a solution* (v_2,χ_2), *square integrable in* ξ, *jointly smooth in* (z,τ), *with* v_2 *going to zero as* $z\to+\infty$. *Then* $\widehat{w}(\cdot,\tau)=\mathcal{F}[w(\cdot,\tau)]$ *satisfies a nonlocal equation of the form*

$$a_0(k)\,\partial_\tau\widehat{w}(k,\tau) \;+\; \int_{-\infty}^{+\infty} a_1(k-\ell,\ell)\;\widehat{w}(k-\ell,\tau)\widehat{w}(\ell,\tau)\,\mathrm{d}\ell \;=\; 0\,, \tag{2.20}$$

where a_0 *and* a_1 *are given by* (2.24) *and* (2.25) *below.*

Proof. By Fourier transform in ξ, (2.16) becomes

$$\left\{\begin{array}{lcll} k\,\mathbb{A}(\underline{u},\mathrm{i}\eta)\,\widehat{v_2} \,+\, \breve{\mathbb{A}}^d(\underline{u})\,\partial_z\widehat{v_2} + m_1 & = & 0_{2n}\,, & z>0\,,\\[2ex] \mathrm{i}\,k\,\widehat{\chi_2}\,J\eta \,+\, H(\underline{u})\cdot\widehat{v_2} + g_1 & = & 0_{n+p}\,, & z=0\,,\end{array}\right. \tag{2.21}$$

where

$$m_1 \;:=\mathcal{F}[\mathcal{M}(\underline{u},\eta;v_1,\partial_\xi\chi_1)\cdot v_1]\,,\quad g_1 \;:=\; \mathcal{F}[G(\underline{u},\eta;v_1,\partial_\tau\chi_1)]\,.$$

A crucial fact in what follows on will be that $m_1(k,z,\tau)$ decays exponentially fast to zero as z goes to $+\infty$, as v_1 itself.

We first eliminate $\widehat{\chi_2}$ from the boundary condition in (2.21), as we have made for $\widehat{\chi_1}$ in (2.17). This yields

$$\widehat{\chi_2}(k) \;=\; \frac{\mathrm{i}}{k}\,\left(H^1(\underline{u})\,\widehat{v_2}(k,0,\tau) \,+\, g_1^1(k,\tau)\right)\,,$$

where g_1^1 denotes the first component of g_1, and

$$\mathcal{C}(\underline{u},\eta)\,\widehat{v_2}(k,0,\tau) \,+\, T(\eta)\,g_1(k,\tau) \;=\; 0\,. \tag{2.22}$$

Now, decomposing $\widehat{v_2}$ as

$$\widehat{v_2}(k,z,\tau) \;=\; \sum_{j=1}^{q_-} \nu_j^-(k,z,\tau)\, R_j^- \;+\; \sum_{j=1}^{q_+} \nu_j^+(k,z,\tau)\, R_j^+\,,$$

thanks to the normalization of left and right eigenvectors, we see that the first equation in (2.21) is equivalent to

$$\partial_z \nu_j^\pm \;-\; k\,\beta_j^\pm \;+\; (L_j^\pm)^*\, m_1 \;=\; 0\,,\quad j\in\{1,\dots,q_\pm\}\,.$$

Solving these ODEs, taking into account the signs of $\mathrm{Re}(\beta_j^\pm)$ and the fact that m_1 decays exponentially fast to zero as z goes to $+\infty$, we find that for $\widehat{v_2}$ to decay to zero as z goes to $+\infty$, necessarily

$$\nu_j^-(k,0,\tau) = \int_0^{+\infty} \mathrm{e}^{-k\beta_j^- z}\,(L_j^-)^*\, m_1(k,z,\tau)\,\mathrm{d}z$$

for ($k<0$ and $j\in\{1,\dots,q_-\}$) or ($k>0$ and $j\in\{m+1,\dots,q_-\}$), and

$$\nu_j^+(k,0,\tau) = \int_0^{+\infty} \mathrm{e}^{-k\beta_j^+ z}\,(L_j^+)^*\, m_1(k,z,\tau)\,\mathrm{d}z$$

for ($k>0$ and $j\in\{1\dots q_+\}$) or ($k<0$ and $j\in\{m+1\dots q_+\}$).

Going back to the boundary condition in (2.22) and multiplying it successively by σ^* and $\overline{\sigma^*}$, we get, thanks to (2.13) and (2.14),

$$\sum_{j=1}^{q_+} (\sigma^*\,\mathcal{C}(\underline{u},\eta)\,R_j^+)\,\nu_j^+(k,0,\tau) \;+\; \sigma^*\,T(\eta) g_1(k,\tau) \;=\; 0\,,$$

$$\sum_{j=1}^{m} (\overline{\sigma}^*\,\mathcal{C}(\underline{u},\eta)\,R_j^-)\,\nu_j^-(k,0,\tau) \;+\; \sum_{j=m+1}^{q_+} (\overline{\sigma}^*\,\mathcal{C}(\underline{u},\eta)\,R_j^+)\,\nu_j^+(k,0,\tau)$$
$$+\,\overline{\sigma}^*\,T(\eta) g_1(k,\tau) \;=\; 0\,.$$

Substituting the integrals found above for $\nu_j^\pm(k,0,\tau)$, we obtain for all $k\neq 0$,

$$\int_0^{+\infty} L(k,z)\, m_1(k,z,\tau)\,\mathrm{d}z \;+\; \sigma(k)\,T(\eta) g_1(k,\tau) \;=\; 0\,, \tag{2.23}$$

with $\sigma(k) := \overline{\sigma}^*$ for $k<0$ and $\sigma(k) := \sigma^*$ for $k>0$,

$$L(k,z) \;:=\; \sum_{j=1}^{q_+} (\sigma^* \mathcal{C}(\underline{u},\eta)\,R_j^+)\,\mathrm{e}^{-k\beta_j^+ z}\,(L_j^+)^*\,,\quad k>0\,,$$

$$L(k,z) \;:=\; \overline{L(-k,z)} \;=\; \sum_{j=1}^{m} (\overline{\sigma}^* \mathcal{C}(\underline{u},\eta)\,R_j^-)\,\mathrm{e}^{-k\beta_j^- z}\,(L_j^-)^*$$
$$+\sum_{j=m+1}^{q_+} (\overline{\sigma}^* \mathcal{C}(\underline{u},\eta)\,R_j^+)\mathrm{e}^{-k\beta_j^+ z}\,(L_j^+)^*\,,\; k<0\,.$$

Finally, the compatibility equation (2.23) may be rewritten explicitely in terms of the amplitude function $\widehat{w} = \widehat{w}(k,\tau)$ and of the linear surface wave function $\widehat{r} = \widehat{r}(k,z)$ (given by Proposition 2.1). Indeed, substituting $\widehat{w}(k,\tau)\,\widehat{r}(k,z)$ for $\widehat{v_1}(k,z,\tau)$ in the definition of m_1, we get

$$m_1(k,z,\tau) \;=\; \partial_\tau \widehat{w}(k,\tau)\,\mathbb{A}^0(\underline{u})\widehat{r}(k,z) + \int_{-\infty}^{+\infty} m(\underline{u},\eta;k-\ell,\ell,z)\,\widehat{w}(k-\ell,\tau)\,\widehat{w}(\ell,\tau)\;\mathrm{d}\ell\,,$$

$$\begin{aligned} 2\pi\cdot m(\underline{u},\eta;k,\ell,z) \;=\;\; & \mathrm{i}\;\ell\,\mathrm{d}\mathbb{A}(\underline{u},\eta)\cdot\widehat{r}(k,z)\cdot\widehat{r}(\ell,z) \\ & +\mathrm{i}\;\ell\,\mathrm{d}\mathbb{A}^d(\underline{u})\cdot\widehat{r}(k,z)\cdot\mathcal{F}[r'](\ell,z) \\ & +\mathrm{i}\;\ell\,H^1(\underline{u})\widehat{r}(k,0)\;\mathbb{A}(\underline{u},\eta)\mathcal{F}[r'](\ell,z)\,, \end{aligned}$$

where we have introduced a new vector-valued function r', defined by

$$\mathrm{i}\;\ell\;\mathcal{F}[r'](\ell,z) \;:=\; \breve{\mathbb{I}}_{2n}\;\partial_z\widehat{r}(\ell,z)\,,$$

or equivalently,

$$\mathcal{F}[r'](k,z) \;=\; \begin{cases} -\mathrm{i}\;\breve{\mathbb{I}}_{2n}\;\displaystyle\sum_{j=1}^{m}\gamma_j\,\beta_j^-\;\mathrm{e}^{\beta_j^-\,k\,z}\;R_j^-\,, & k\;>\;0\,,\,z>0\,,\\ -\mathrm{i}\;\breve{\mathbb{I}}_{2n}\;\displaystyle\sum_{j=1}^{m}\overline{\gamma_j}\,\beta_j^+\;\mathrm{e}^{\beta_j^+\,k\,z}\;R_j^+\,, & k\;<\;0\,,z>0\,. \end{cases}$$

To find the last term in the kernel m, we have used the expression of $\widehat{\chi_1}$ given by (2.18). This expression is also useful to compute

$$\begin{aligned} g_1(k,\tau) \;=\;\; & \tfrac{\mathrm{i}}{k}\,\partial_\tau\widehat{w}(k,\tau)H^1(\underline{u})\,\widehat{r}(k,0)\,\mathbf{e}_1 \\ & +\int_{-\infty}^{+\infty} g(\underline{u},\eta;k-\ell,\ell)\,\widehat{w}(k-\ell,\tau)\,\widehat{w}(\ell,\tau)\;\mathrm{d}\ell\,, \end{aligned}$$

$$g(\underline{u},\eta;k,\ell) \;:=\; \frac{1}{4\pi}\,\mathrm{d}^2h(\underline{u})\cdot(\widehat{r}(k,0),\widehat{r}(\ell,0))\,.$$

We have thus obtained the nonlocal equation (2.20) for $\widehat{w}$, with

$$a_0(k) := \int_0^{+\infty} L(k,z)\,\mathbb{A}^0(\underline{u})\,\widehat{r}(k,z)\,\mathrm{d}z \;+\; \frac{\mathrm{i}}{k}\,(H^1(\underline{u})\widehat{r}(k,0))\,\sigma(k)\,T(\eta)\,\mathbf{e}_1\,, \tag{2.24}$$

$$a_1(k,\ell) := \int_0^{+\infty} L(k+\ell,z)\,m(\underline{u},\eta;k,\ell,z)\,\mathrm{d}z \;+\; \sigma(k+\ell)\,T(\eta)\,g(\underline{u},\eta;k,\ell)\,. \tag{2.25}$$

□

Remark 2.3 *Since $L(k,z)$ and $\widehat{r}(k,z)$ are linear combinations of exponentials* $\mathrm{e}^{-k\beta_j^+ z}$ *and* $\mathrm{e}^{k\beta_p^- z}$*, and, by construction,* $L(k,z) = \overline{L(-k,z)}$*,* $\widehat{r}(k,z) = \overline{\widehat{r}(-k,z)}$*,* $\sigma(k) = \overline{\sigma(-k)}$*, we see on* (2.24) *that* a_0 *is of the form*

$$a_0(k) = \begin{cases} \frac{\alpha_0}{k}\,, & k > 0\,, \\ -\frac{\overline{\alpha_0}}{k}\,, & k < 0\,, \end{cases} \tag{2.26}$$

with $\alpha_0 \in \mathbb{C}$*. More explicitly, this number is given by*

$$\alpha_0 \;=\; \frac{\sigma^* \mathcal{C}(\underline{u})\, R_j^+}{\beta_j^+ - \beta_p^-}\,(L_j^+)^* \mathbb{A}^0(\underline{u})(\gamma_p R_p^-) \;+\; \mathrm{i}\,(\gamma_p H^1(\underline{u}) R_p^-)\,\sigma^* T(\eta)\,\mathbf{e}_1\,, \tag{2.27}$$

where we have used the usual summation convention on the repeated indices, with $j \in \{1, \ldots, q_+\}$*,* $p \in \{1, \ldots, m\}$*.*

Remark 2.4 *The kernel* a_1 *is obviously not symmetric in* (k,ℓ)*. However, it can easily be symmetrized. Indeed, by change of variables* $\ell \mapsto k - \ell$*, the nonlocal equation* (2.20) *is equivalent to*

$$a_0(k)\,\partial_\tau \widehat{w}(k,\tau) \;+\; \int_{-\infty}^{+\infty} a_1^s(k-\ell,\ell)\;\widehat{w}(k-\ell,\tau)\widehat{w}(\ell,\tau)\,\mathrm{d}\ell \;=\; 0\,,$$

with

$$a_1^s(k,\ell) \;:=\; \int_0^{+\infty} L(k+\ell,z)\, m^s(\underline{u},\eta;k,\ell,z)\,\mathrm{d}z \;+\; \sigma(k+\ell)\,T(\eta)\,g(\underline{u},\eta;k,\ell)\,, \tag{2.28}$$

$$\begin{aligned} 2\,m^s(\underline{u},\eta;k,\ell,z) \;:=\; & \mathrm{i}(k+\ell)\,\mathrm{d}\mathbb{A}(\underline{u},\eta)\cdot\widehat{r}(k,z)\cdot\widehat{r}(\ell,z) \\ & + \mathrm{i}\ell\,\mathrm{d}\mathbb{A}^d(\underline{u})\cdot\widehat{r}(k,z)\cdot\mathcal{F}[r'](\ell,z) \;+\; \mathrm{i}k\,\mathrm{d}\mathbb{A}^d(\underline{u})\cdot\widehat{r}(\ell,z)\cdot\mathcal{F}[r'](k,z) \\ & + \mathrm{i}\ell\, H^1(\underline{u})\widehat{r}(k,0)\;\mathbb{A}(\underline{u},\eta)\mathcal{F}[r'](\ell,z) \;+\; \mathrm{i}k\, H^1(\underline{u})\widehat{r}(\ell,0)\;\mathbb{A}(\underline{u},\eta)\mathcal{F}[r'](k,z)\,. \end{aligned} \tag{2.29}$$

(The first term is indeed symmetric by the symmetry of $\mathrm{d}\mathbb{A}$*, as a linear combination of second order differentials* $\mathrm{d}^2 f^j$*. For the same reason,* g *being defined by means of* $\mathrm{d}^2 h$*, it is symmetric in* (k,ℓ)*.) In addition, both the integral and the last term in* $a_1^s(k,\ell)$ *are* positively homogeneous degree zero *in* (k,ℓ)*.*

Theorem 2.5 *Under the Assumptions 1, 2, 3, 4, 5, 6, we also assume that the number* α_0 *defined in* (2.27) *is nonzero. Then weakly nonlinear surface waves for the nonlinear model* (2.5) *are governed by the nonlocal amplitude equation*

$$\partial_\tau w + \partial_\xi Q\,[w] = 0\,, \tag{2.30}$$

where Q is given by

$$Q\,[v]\,(\xi) = (K * (v \otimes v))(\xi, \xi) \tag{2.31}$$

for all $v \in \mathscr{S}(\mathbb{R})$ (the Schwartz class), the kernel K being the real tempered distribution on $\mathbb{R}^2$

$$K := 2\pi \mathcal{F}^{-1}(\Lambda)\,, \tag{2.32}$$

with $\Lambda \in L^\infty(\mathbb{R}^2)$ defined as in (2.33).

Proof. With the notations introduced above, we define for $k \neq 0$, $\ell \neq 0$, $k + \ell \neq 0$,

$$\Lambda(k, \ell) \;:=\; \frac{a_1^s(k, \ell)}{\mathrm{i}\;(k+\ell)\,a_0(k+\ell)}\,. \tag{2.33}$$

Then (2.20) can be rewritten as

$$\partial_\tau \widehat{w}(k, \tau) + \mathrm{i}k \int_{-\infty}^{+\infty} \Lambda(k-\ell, \ell)\widehat{w}(k-\ell, \tau)\widehat{w}(\ell, \tau)\mathrm{d}\ell = 0\,. \tag{2.34}$$

By inverse Fourier transform this gives (2.30) with, *formally*,

$$Q\,[w]\,(\xi, \tau) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}(k+\ell)\xi} \Lambda(k, \ell)\widehat{w}(k, \tau)\widehat{w}(\ell, \tau)\,\mathrm{d}\ell\,\mathrm{d}k\,,$$

or,

$$Q\,[w]\,(\xi, \tau) = 2\pi \mathcal{F}^{-1}(\Lambda\;\widehat{w}(\cdot, \tau) \otimes \widehat{w}(\cdot, \tau))(\xi, \xi)\,,$$

where $\mathcal{F}$ here denotes the Fourier transform on $\mathscr{S}'(\mathbb{R}^2)$. Since $\mathcal{F}^{-1}(\widehat{w} \otimes \widehat{w}) = w \otimes w$, we find that

$$2\pi \mathcal{F}^{-1}(\Lambda \widehat{w}(\cdot, \tau) \otimes \widehat{w}(\cdot, \tau)) \;=\; K * (w(\cdot, \tau) \otimes w(\cdot, \tau))\,,$$

with $K := 2\pi \mathcal{F}^{-1}(\Lambda)$.

To justify the above computations, we first observe that the kernel Λ has some nice properties inherited from the properties of a_1^s and a_0. It is indeed smooth (analytic) outside the lines $k = 0$, $\ell = 0$, and $k + \ell = 0$, *symmetric* in (k, l), like a_1^s, and positively *homogeneous* degree zero, like a_1^s and $k \mapsto ka_0(k)$. In addition, since $\overline{a_0(-k)} = a_0(k)$ and $\overline{a_1^s(-k, -\ell)} = a_1^s(k, \ell)$, we have $\overline{\Lambda(-k, -\ell)} = \Lambda(k, \ell)$. To summarize, we have for all $k \neq 0$, $\ell \neq 0$, and $\theta > 0$,

$$\Lambda(k, \ell) \;=\; \Lambda(\ell, k)\,,\; \Lambda(-k, -\ell) \;=\; \overline{\Lambda(k, \ell)}\,,\; \Lambda(\theta k, \theta \ell) \;=\; \Lambda(k, \ell)\,. \tag{2.35}$$

Using these properties and noting that $\Lambda(1, \theta)$ and $\Lambda(-1, \theta)$ are uniformly bounded for $\theta \in (0, 1)$, we easily check that Λ is bounded on $(\mathbb{R}\backslash\{0\})^2$. Thus it can be viewed as a tempered distribution, and K is therefore well-defined by (2.32) as a tempered distribution. Furthermore, the second property in (2.35) shows that K is a *real* distribution.

To conclude, for all $v \in L^2(\mathbb{R})$,

$$k \mapsto \int_{-\infty}^{+\infty} \Lambda(k-\ell,\ell)\widehat{v}(k-\ell)\widehat{v}(\ell)\mathrm{d}\ell$$

defines an L^∞ function by the Cauchy-Schwarz inequality, whose inverse Fourier transform, $Q[v]$, is a tempered distribution. If moreover v belongs to the Schwartz class, $Q[v]$ is a function, explicitly given in terms of K by (2.31). □

In [10], Hunter had pointed out the following stability condition for equations of the form (2.34) with Λ satisfying (2.35),

$$\Lambda(1,0+) = \overline{\Lambda(1,0-)}\,. \tag{2.36}$$

He had in particular checked it was satisfied in the case of weakly nonlinear surfaces waves in Elasticity [10, 14, 15]. More recently, he and co-workers derived and investigated a stronger condition [2, 1], which ensures that (2.34) has a Hamiltonian structure (see [11] for a local-in-time existence under this condition in a periodic setting). It turns out that (2.36) is in fact exactly what we need to get a priori estimates without loss of derivatives for (2.34), see [4]. This is the condition we are going to investigate further in our abstract framework and afterwards in the explicit case of subsonic phase boundaries.

Proposition 2.6 *For Λ defined as in* (2.33) *with a_0 given by* (2.26) (2.27)*, α_0 being assumed to be nonzero, and a_1^s given by* (2.28) (2.29)*, the stability condition* (2.36) *is equivalent to requiring that $a(P)$ and $a(Q)$ be real, with a the linear form $a : \mathbb{C}^{2n} \to \mathbb{C}$ defined by*

$$\alpha_0\, a(R) \;=\; -\mathrm{i}\sigma^* \underline{T}\, \underline{\mathrm{d}^2 h} \cdot (R, V) \;+$$

$$\sum_{j=1}^{q_+} \sum_{p=1}^{m} \frac{\sigma^* \underline{\mathcal{C}}\, R_j^+}{\beta_j^+ - \beta_p^-} \,(L_j^+)^* \Big((\underline{\mathrm{d}\mathbb{A}} - \mathrm{i}\beta_p^- \underline{\mathrm{d}\breve{\mathbb{A}}}^d) \cdot (\gamma_p R_p^-) \cdot R - \mathrm{i}\beta_p^- (\underline{H}^1 R) \underline{\breve{\mathbb{A}}}(\gamma_p R_p^-) \Big)\;,$$

where, for simplicity, underlined letters correspond to quantities evaluated at $\underline{u}$ and/or η, while, as in Proposition 2.1,

$$V \;=\; \sum_{p=1}^{m} \gamma_p\, R_p^-\;, \quad P \;=\; \mathrm{Re}\Big(\sum_{p=1}^{m} \gamma_p\, R_p^- \Big)\;, \quad Q \;=\; \mathrm{Im}\Big(\sum_{p=1}^{m} \gamma_p\, R_p^- \Big)\;.$$

Proof. By direct computation we find that

$$2\alpha_0\,\Lambda(1,0+) = -\frac{\mathrm{i}}{2\pi}\,\sigma^*\underline{T}\,\underline{\mathrm{d}^2 h}\cdot(V,V)$$

$$+\frac{\sigma^*\underline{\mathcal{C}}\,R_j^+}{\beta_j^+-\beta_p^-}\,(L_j^+)^*\Big((\underline{\mathrm{d}\mathbb{A}}-\mathrm{i}\beta_p^-\underline{\mathrm{d}\breve{\mathbb{A}}}^d)\cdot(\gamma_p R_p^-)\cdot V-\mathrm{i}\beta_p^-(\underline{H}^1V)\underline{\breve{\mathbb{A}}}(\gamma_p R_p^-)\Big)\ ,$$

$$2\alpha_0\,\Lambda(1,0-) = -\frac{\mathrm{i}}{2\pi}\,\sigma^*\underline{T}\,\underline{\mathrm{d}^2 h}\cdot(\overline{V},V)$$

$$+\frac{\sigma^*\underline{\mathcal{C}}\,R_j^+}{\beta_j^+-\beta_p^-}\,(L_j^+)^*\Big((\underline{\mathrm{d}\mathbb{A}}-\mathrm{i}\beta_p^-\underline{\mathrm{d}\breve{\mathbb{A}}}^d)\cdot(\gamma_p R_p^-)\cdot \overline{V}-\mathrm{i}\beta_p^-(\underline{H}^1\overline{V})\underline{\breve{\mathbb{A}}}(\gamma_p R_p^-)\Big)\ ,$$

where for simplicity we have used the convention of summation over repeated indices. Observe that (2.36) is equivalent to require that $\Lambda(1,0+)+\Lambda(1,0-)\in\mathbb{R}$ and $\Lambda(1,0+)-\Lambda(1,0-)\in\mathrm{i}\mathbb{R}$, or that the sum of the above equalities divided by $2\alpha_0$ and their difference divided by $2\mathrm{i}\alpha_0$ must be real. □

3 Application to van der Waals fluids

In this section we apply the method of the previous section to a concrete model for fluids exhibiting phase changes, and obtain an explicit form for the kernel as in Theorem 2.5.

3.1 Introduction

The Euler equations governing the motion in $\mathbb{R}^d$, $d\geq 1$, of a compressible, non-viscous, isothermal fluid of van der Waals are

$$\begin{cases} \partial_t\rho+\nabla\cdot(\rho v)=0_d \\ \partial_t(\rho v)+\nabla\cdot(\rho v\otimes v)+\nabla p=0\ . \end{cases} \tag{3.37}$$

Above $\rho>0$ denotes the density, $v\in\mathbb{R}^d$ the velocity and $p>0$ the pressure of the fluid obeying the pressure law

$$p(V)=\frac{RT}{V-b}-\frac{a}{V^2}\ ,$$

where $V := 1/\rho$ is the specific volume, T is the temperature, R is the perfect gas constant and a, b are positive constants. Below the critical temperature, $T_c := 8a/(27bR)$, van der Waals fluids can undergo transitions between two phases, the liquid phase for $1/\rho\in(0,V_*)$ and the vapor phase for $1/\rho\in(V^*,\infty)$, for the presence of the nonphysical region (V_*,V^*), called the *spinodal region.* The van der Waals law is considered here for concreteness, but our results do not depend on the actual form of this law. They basically depend on the existence of three zones, namely the intervals $(0, V_*)$ and (V^*, ∞) where the pressure is decreasing with $1/\rho$ and the system (3.37) is hyperbolic, and the interval (V_*, V^*) where the pressure is increasing with $1/\rho$ and the system (3.37) becomes elliptic. In this situation it is natural to consider (weak) solutions to (3.37) that avoid the spinodal region. The simplest weak solutions in this case are piecewise $\mathscr{C}^1$ functions which satisfy (3.37) outside a moving interface $\Sigma(t)$, and, at least, the Rankine-Hugoniot jump conditions across the interface. We are interested here in *dynamic* discontinuities, for which there is some mass tranfer across the interface, and especially the *subsonic* ones, for which the Mach numbers with respect to the interface are lower than one on both sides. In the terminology of hyperbolic conservation laws these discontinuities are *undercompressive*, the number of outgoing characteristics being equal to that of incoming ones, and an additional jump condition is thus needed. In the continuation of [3], we have chosen a simple and explicit additional condition, referred to as the capillarity criterion merely because it is equivalent to the existence of travelling capillarity profiles. It can be understood as the conservation of 'total energy', namely the kinetic energy plus the free energy, across the interface. It amounts to neglecting dissipation due to viscosity, which is reasonable in some physicals contexts (e.g. for water in extreme conditions or for superfluids).

3.2 The nonlinear problem and the reference phase boundary

We consider a problem of the form (2.6) with $n = d + 1$, $p = 1$, and

$$u := \begin{pmatrix} \rho \\ \mathbf{J} \end{pmatrix}, \quad f^0(u) := u, \quad f^i(u) := \begin{pmatrix} \mathbf{J}_i \\ p(\rho)\,\mathbf{e}_i + \frac{\mathbf{J}_i}{\rho}\,\mathbf{J} \end{pmatrix},$$

$$\widetilde{f}^0(u) := \begin{pmatrix} u \\ \frac{\|\mathbf{J}\|^2}{2\rho} + \rho\, f(\rho) \end{pmatrix}, \quad \widetilde{f}^i(u) := \begin{pmatrix} f^i(u) \\ \left(\frac{\|\mathbf{J}\|^2}{2\rho^2} + f(\rho) + \frac{p(\rho)}{\rho}\right)\mathbf{J}_i \end{pmatrix},$$

where $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ is the canonical basis of $\mathbb{R}^d$,

$$\mathbf{J} = (\mathbf{J}_1, \dots, \mathbf{J}_d) := \rho\, \mathbf{v} \in \mathbb{R}^{d+1}$$

is the momentum, and $f = f(\rho)$ is the *free specific energy* of the fluid, characterized by

$$p(\rho) = \rho^2\, f'(\rho). \tag{3.38}$$

By definition, a solution of (2.6) with $\chi \equiv 0$ and u constant on either side of the hyperplane $\{x \in \mathbb{R}^d :\ x_d = 0\}$ is characterized by $[\widetilde{f}^d(u)] = 0$, that is,

$$[\mathbf{J}_d] = 0\,, \quad \left[p(\rho)\,\mathbf{e}_d + \frac{\mathbf{J}_d}{\rho}\,\mathbf{J}\right] = 0\,, \quad \left[\left(\frac{\|\mathbf{J}\|^2}{2\rho^2} + f(\rho) + \frac{p(\rho)}{\rho}\right)\mathbf{J}_d\right] = 0\,. \tag{3.39}$$

As is very well-known, the first two equations imply that for a *dynamical* discontinuity, for which $\mathbf{J}_d \neq 0$, the jump of the tangential velocity must be zero, that is, $[\mathbf{v}_1] = \cdots = [\mathbf{v}_{d-1}] = 0$. By a change of Galilean frame we may assume without loss of generality that the tangential velocity of the left and right reference states is zero. With this simplification, the jump conditions in (3.39) reduce to

$$[\mathbf{J}_d] = 0\,, \quad \left[p(\rho) + \frac{\mathbf{J}_d^2}{\rho}\right] = 0\,, \quad \left[\frac{\mathbf{J}_d^2}{2\rho^2} + f(\rho) + \frac{p(\rho)}{\rho}\right] = 0\,. \tag{3.40}$$

For later use, it is convenient to introduce the functions

$$q : (\rho, j) \mapsto p(\rho) + \frac{j^2}{\rho}\,,$$

$$z : (\rho, j) \mapsto j\left(\frac{j^2}{2\rho^2} + f(\rho) + \frac{p(\rho)}{\rho}\right)\,.$$

Notice that using these functions the jump conditions in (3.40) equivalently read

$$[\mathbf{J}_d] = 0\,, \quad [q(\rho, \mathbf{J}_d)] = 0\,, \quad [z(\rho, \mathbf{J}_d)] = 0\,. \tag{3.41}$$

It is not difficult to show that for a non-monotone pressure law $\rho \mapsto p(\rho)$, there exist ρ_l, ρ_r, v_l, v_r satisfying (3.41) with $\rho_l v_l = \rho_r v_r =: \mathbf{J}_d > 0$, that is, $q(\rho_l, \mathbf{J}_d) = q(\rho_r, \mathbf{J}_d)$ and $z(\rho_l, \mathbf{J}_d) = z(\rho_r, \mathbf{J}_d)$, together with $q_\rho(\rho_{l,r}, \mathbf{J}_d) \neq 0$, that is $p'(\rho_{l,r}) - v_{l,r}^2 \neq 0$; see [3, page 249]. The corresponding reference states $u_l = {}^t(\rho_l, 0, \ldots, 0, v_l)$ and $u_r = {}^t(\rho_r, 0, \ldots, 0, v_r)$ are thus connected by a planar dynamical subsonic phase boundary located at $x_d = 0$, and $\underline{u} = (u_l, u_r)$, satisfies Assumption 1. From now on, we fix $\underline{u}$ as above, and we introduce the notations $c_{l,r}$ for the sound speeds on each side of the reference phase boundary:

$$c_{l,r} := \sqrt{p'(\rho_{l,r})}\,.$$

3.3 Linearization

Proceeding as in Section 2.2, we may reformulate the free boundary problem (2.5) with our specific fluxes in terms of

$$\rho_\pm(y_0, y_1, \ldots, y_d) := \rho(y_0, \ldots, y_{d-1}, \chi(y_0, \ldots, y_{d-1}) \pm y_d)\,,$$

$$\mathbf{J}_\pm(y_0, y_1, \ldots, y_d) := \mathbf{J}(y_0, \ldots, y_{d-1}, \chi(y_0, \ldots, y_{d-1}) \pm y_d)\,,$$

as

$$\left\{\begin{array}{lcll} \mathbb{L}(\rho_\pm, \mathbf{J}_\pm, \nabla\chi) \cdot \begin{pmatrix} \rho_- \\ \mathbf{J}_- \\ \rho_+ \\ \mathbf{J}_+ \end{pmatrix} & = & 0_{2d+2} & , \quad y_d > 0 \\ b(\rho_\pm, \mathbf{J}_\pm, \nabla\chi) & = & 0_{d+2} & , \quad y_d = 0\,. \end{array}\right. \tag{3.42}$$

Linearizing this problem about $(\rho_- \equiv \rho_l, \mathbf{j}_- \equiv (0,\dots,0,\rho_l v_l), \rho_+ \equiv \rho_r, \mathbf{j}_+ \equiv (0,\dots,0,\rho_r v_r), \chi \equiv 0)$, we readily get a system of the form (2.9), without having to invoke Assumption 2 for the reduction of the boundary conditions. Indeed, for the specifix fluxes we are considering, the linearized version of the jump conditions in (2.6) turns out to reduce to

$$\left\{\begin{array}{l} [\rho]\,\partial_t\dot{\chi} \,=\, [\mathbf{j}_d]\,, \\[2ex] [p]\,\partial_i\dot{\chi} \,=\, [v\,\mathbf{j}_i]\,,\ i\in\{1,\dots,d-1\}\,, \\[2ex] \big[(c^2-v^2)\,\dot{\rho} \,+\, 2\,v\,\mathbf{j}_d\big] \,=\, 0\,, \\[2ex] \big[\tfrac{1}{2}\,\rho\,v^2 \,+\, \rho\,f\big]\,\partial_t\dot{\chi} \,=\, \Big[(c^2-v^2)v\dot{\rho} \,+\, (f+\tfrac{p}{\rho} \,+\, \tfrac{3}{2}v^2)\,\mathbf{j}_d\Big]\,, \end{array}\right. \tag{3.43}$$

which is obviously of the form

$$J(\underline{u})\nabla\dot{\chi} \,+\, H(\underline{u})\cdot(\dot{\rho},\mathbf{j}) \,=\, 0_{d+2}\,, \tag{3.44}$$

with $J(\underline{u})$ a matrix depending only on the reference state, as well as $H(\underline{u})$, and

$$(\dot{\rho},\mathbf{j}) := \begin{pmatrix} \dot{\rho}_- \\ \mathbf{j}_- \\ \dot{\rho}_+ \\ \mathbf{j}_+ \end{pmatrix}.$$

Regarding the linearized version of the interior equation in (2.6), it is given by the block-diagonal operator

$$\mathbb{L}(\rho_{l,r},0,v_{l,r},0,0) = \left(\begin{array}{c|c} \mathbb{L}_-(\rho_l,v_l) & 0 \\ \hline 0 & \mathbb{L}_+(\rho_r,v_r)\,, \end{array}\right),$$

the operators $\mathbb{L}_\pm$ being defined in tangential Fourier variables by

$$\widehat{\mathbb{L}_\pm}(\rho,v,\eta) \,=\, \left(\begin{array}{c|c|c} \mathrm{i}\eta_0 & \mathrm{i}\check{\eta} & \pm\partial_{y_d} \\ \hline \mathrm{i}p'(\rho)\,{}^t\check{\eta} & (\mathrm{i}\eta_0 \pm v\partial_{y_d})\mathrm{I}_{d-1} & 0_{d-1} \\ \hline \pm(p'(\rho)-v^2)\partial_{y_d} & \mathrm{i}v\check{\eta} & \mathrm{i}\eta_0 \pm 2v\partial_{y_d} \end{array}\right),$$

where $\check{\eta} := (\eta_1,\dots,\eta_{d-1})$. The subsonicity of the reference phase boundary $(c_{l,r} \,>\, v_{l,r})$ and the additional assumption

$$\eta_0^2 \,<\, (c_{l,r}^2 - v_{l,r}^2)\,\|\check{\eta}\|^2 \tag{3.45}$$

imply that Assumption 5 is satisfied (see [3]). In this case, with the notations of Section 2.2 we have,

$$n = d+1,\ p = 1,\ q_- = q_+ = d+1,\ m = 2\,,$$

the eigenvalues $\beta_1^\pm$ being the roots of

$$(c_l^2 - v_l^2)\beta^2 \,+\, 2\,\mathrm{i}\eta_0 v_l \beta \,+\, \eta_0^2 - c_l^2\|\check{\eta}\|^2 \,=\, 0\,,$$

and the eigenvalues $\beta_2^\pm$ being the roots of

$$(c_r^2 - v_r^2)\beta^2 \,-\, 2\,\mathrm{i}\eta_0 v_l \beta \,+\, \eta_0^2 - c_r^2\|\check{\eta}\|^2 \,=\, 0\,,$$

which gives explicitly,

$$\begin{aligned}
&\beta_1^+ \,=\, \tfrac{1}{c_l^2-v_l^2}\,(\alpha_l - \mathrm{i}\eta_0 v_l)\,, \quad \beta_1^- \,=\, \tfrac{1}{c_l^2-v_l^2}\,(-\alpha_l - \mathrm{i}\eta_0 v_l)\,,\\
&\alpha_l \,:=\, c_l\,\sqrt{(c_l^2 - v_l^2)\|\check{\eta}\|^2 - \eta_0^2}\,,\\
&\beta_2^+ \,=\, \tfrac{1}{c_r^2-v_r^2}\,(\alpha_r + \mathrm{i}\eta_0 v_r)\,, \quad \beta_2^- \,=\, \tfrac{1}{c_r^2-v_r^2}\,(-\alpha_r + \mathrm{i}\eta_0 v_r)\,,\\
&\alpha_r \,:=\, c_r\,\sqrt{(c_r^2 - v_r^2)\|\check{\eta}\|^2 - \eta_0^2}\,.
\end{aligned} \tag{3.46}$$

The other, purely imaginary eigenvalues are

$$\beta_3^+ \,:=\, \mathrm{i}\frac{\eta_0}{v_l}\,, \quad \beta_3^- \,:=\, -\mathrm{i}\frac{\eta_0}{v_r}\,,$$

of multiplicity $d-1$. Right eigenvectors may be chosen as follows

$$\begin{array}{llll}
R_1^\pm = \begin{pmatrix} \mathrm{r}_1^\pm \\ 0_{d+1} \end{pmatrix}, & & R_2^\pm = \begin{pmatrix} 0_{d+1} \\ \mathrm{r}_2^\pm \end{pmatrix}, & \\
\mathrm{r}_1^- \;=\; \begin{pmatrix} -\mathrm{i}\eta_0 + v_l\beta_1^- \\ \mathrm{i}c_l^2\check{\eta} \\ \alpha_l \end{pmatrix}, & & \mathrm{r}_1^+ \;=\; \begin{pmatrix} \mathrm{i}\eta_0 - v_l\beta_1^+ \\ -\mathrm{i}c_l^2\check{\eta} \\ \alpha_l \end{pmatrix}, & \\
\mathrm{r}_2^- \;=\; \begin{pmatrix} -\mathrm{i}\eta_0 - v_r\beta_2^- \\ \mathrm{i}c_r^2\check{\eta} \\ -\alpha_r \end{pmatrix}, & & \mathrm{r}_2^+ \;=\; \begin{pmatrix} \mathrm{i}\eta_0 + v_r\beta_2^+ \\ -\mathrm{i}c_r^2\check{\eta} \\ -\alpha_r \end{pmatrix}, & \\
R_i^+ \;=\; \begin{pmatrix} \mathrm{r}_i^+ \\ 0_{d+1} \end{pmatrix}, & & R_i^- \;=\; \begin{pmatrix} 0_{d+1} \\ \mathrm{r}_i^- \end{pmatrix}, & \\
\mathrm{r}_i^+ \;=\; \begin{pmatrix} 0 \\ \eta_0\,\check{\mathrm{e}}_{i-2} \\ v_l\,\check{\eta}\cdot\check{\mathrm{e}}_{i-2} \end{pmatrix}, & & \mathrm{r}_i^- \;=\; \begin{pmatrix} 0 \\ \eta_0\,\check{\mathrm{e}}_{i-2} \\ v_r\,\check{\eta}\cdot\check{\mathrm{e}}_{i-2} \end{pmatrix}, & \\
 & & i = 3,\ldots,d+1\,, &
\end{array} \tag{3.47}$$

where $(\check{\mathrm{e}}_1, \ldots, \check{\mathrm{e}}_{d-1})$ is an arbitrary basis of $\mathbb{R}^{d-1}$. For left eigenvectors, we may take

$$\begin{aligned}
L_1^{\pm} &= \begin{pmatrix} \mathrm{l}_1^{\pm} \\ 0_{d+1} \end{pmatrix}, & L_2^{\pm} &= \begin{pmatrix} 0_{d+1} \\ \mathrm{l}_2^{\pm} \end{pmatrix}, \\
\mathrm{l}_1^{-} &= \begin{pmatrix} \mathrm{i}\eta_0 + 2v_l\overline{\beta_1^-} \\ -\mathrm{i}\check{\eta} \\ -\overline{\beta_1^-} \end{pmatrix}, & \mathrm{l}_1^{+} &= \begin{pmatrix} -\mathrm{i}\eta_0 - 2v_l\overline{\beta_1^+} \\ \mathrm{i}\check{\eta} \\ \overline{\beta_1^+} \end{pmatrix}, \\
\mathrm{l}_2^{-} &= \begin{pmatrix} -\mathrm{i}\eta_0 + 2v_r\overline{\beta_2^-} \\ \mathrm{i}\check{\eta} \\ -\overline{\beta_2^-} \end{pmatrix}, & \mathrm{l}_2^{+} &= \begin{pmatrix} \mathrm{i}\eta_0 - 2v_r\overline{\beta_2^+} \\ -\mathrm{i}\check{\eta} \\ \overline{\beta_2^+} \end{pmatrix}, \\
L_i^{+} &= \begin{pmatrix} \mathrm{l}_i^{+} \\ 0_{d+1} \end{pmatrix}, & L_i^{-} &= \begin{pmatrix} 0_{d+1} \\ \mathrm{l}_i^{-} \end{pmatrix}, \\
\mathrm{l}_i^{+} &= \begin{pmatrix} -v_l^2\,\check{\eta}\cdot\check{\mathrm{e}}'_{i-2} \\ \eta_0\,\check{\mathrm{e}}'_{i-2} \\ v_l\,\check{\eta}\cdot\check{\mathrm{e}}'_{i-2} \end{pmatrix}, & \mathrm{l}_i^{-} &= \begin{pmatrix} -v_r^2\,\check{\eta}\cdot\check{\mathrm{e}}'_{i-2} \\ \eta_0\,\check{\mathrm{e}}'_{i-2} \\ v_r\,\check{\eta}\cdot\check{\mathrm{e}}'_{i-2} \end{pmatrix}, \\
& & & i = 3, \ldots, d+1,
\end{aligned} \tag{3.48}$$

where $(\check{\mathrm{e}}'_1, \ldots, \check{\mathrm{e}}'_{d-1})$ is another arbitrary basis of $\mathbb{R}^{d-1}$. Recalling that

$$\breve{\mathbb{A}}^d(\underline{u}) = \left(\begin{array}{c|c} -A^d(\rho_l, v_l) & 0 \\ \hline 0 & A^d(\rho_r, v_r) \end{array}\right),$$

$$A^d(\rho, v) = \left(\begin{array}{c|c|c} 0 & 0^*_{d-1} & 1 \\ \hline 0_{d-1} & v\mathrm{I}_{d-1} & 0_{d-1} \\ \hline p'(\rho) - v^2 & 0^*_{d-1} & 2v \end{array}\right),$$

we easily compute

$$\mp (L_{i+2}^{\pm})^* \breve{\mathbb{A}}^d(\underline{u}) R_{k+2}^{\pm} = v_{l,r}^3\,(\check{\eta}\cdot\check{\mathrm{e}}'_i)\,(\check{\eta}\cdot\check{\mathrm{e}}_k) + v_{l,r}\,\eta_0^2\,(\check{\mathrm{e}}'_i\cdot\check{\mathrm{e}}_k)$$

for $i, k \in \{1, \ldots, d-1\}$. So, even though the eigenvalues $\beta_3^{\pm}$ are non-simple, it is possible to choose the bases $(\check{\mathrm{e}}_1, \ldots, \check{\mathrm{e}}_{d-1})$ and $(\check{\mathrm{e}}'_1, \ldots, \check{\mathrm{e}}'_{d-1})$ to have

$$(L_i^{\pm})^* \breve{\mathbb{A}}^d(\underline{u}) R_k^{\pm} = 0, \; i, k \in \{3, \ldots, d+1\}, \; i \neq k.$$

For instance, we can take

$$\check{\mathrm{e}}_1 = \check{\mathrm{e}}'_1 = \check{\eta},$$

and choose 'dual' bases $(\check{\mathrm{e}}_2, \ldots, \check{\mathrm{e}}_{d-1})$ and $(\check{\mathrm{e}}'_2, \ldots, \check{\mathrm{e}}'_{d-1})$ of $\check{\eta}^{\perp}$. The left and right eigenvectors above are *not* normalized to have $(L_i^{\pm})^* \breve{\mathbb{A}}^d(\underline{u}) R_i^{\pm} = 1$.

Instead, we have

$$
\begin{array}{rcl}
(L_1^\pm)^*\breve{\mathbb{A}}^d(\underline{u})R_1^\pm & = & 2\,c_l^2\,(v_l\|\check\eta\|^2\,-\,i\,\eta_0\,\beta_1^\pm) \;=\; \frac{2\alpha_l}{c_l^2-v_l^2}\,(v_l\alpha_l \mp \mathrm{i}\eta_0 c_l^2)\,,\\
(L_2^\pm)^*\breve{\mathbb{A}}^d(\underline{u})R_2^\pm & = & 2\,c_r^2\,(v_r\|\check\eta\|^2\,+\,i\,\eta_0\,\beta_2^\pm) \;=\; \frac{2\alpha_r}{c_r^2-v_r^2}\,(v_r\alpha_r \pm \mathrm{i}\eta_0 c_r^2)\,,\\
(L_3^+)^*\breve{\mathbb{A}}^d(\underline{u})R_3^+ & = & -\,v_l\,(\eta_0^2\,+\,v_l^2\,\|\check\eta\|^2)\,\|\check\eta\|^2\,,\\
(L_i^+)^*\breve{\mathbb{A}}^d(\underline{u})R_i^+ & = & -\,v_l\,\eta_0^2\,,\ i=4,\ldots,d+1\,,\\
(L_3^-)^*\breve{\mathbb{A}}^d(\underline{u})R_3^- & = & v_r\,(\eta_0^2\,+\,v_r^2\,\|\check\eta\|^2)\,\|\check\eta\|^2\,,\\
(L_i^-)^*\breve{\mathbb{A}}^d(\underline{u})R_i^- & = & v_r\,\eta_0^2\,,\ i=4,\ldots,d+1\,.
\end{array}
\tag{3.49}
$$

In addition, we do have

$$(L_i^\pm)^*\breve{\mathbb{A}}^d(\underline{u})R_k^\mp \;=\; 0\,,\ i,k\in\{1,\ldots,d+1\}\,.$$

Lemma 3.1 *We assume that, as described above, the states* ${}^t(\rho_l,0,\ldots,0,v_l)$ *and* ${}^t(\rho_r,0,\ldots,0,v_r)$ *are connected by a planar dynamical subsonic phase boundary located at* $x_d=0$. *Without loss of generality we assume that the velocities* v_l *and* v_r *are positive. The associated right eigenvectors are defined as in* (3.47). *Then a linear combination of the form*

$$\begin{pmatrix}\dot\rho_-\\ \mathbf{j}_-\\ \dot\rho_+\\ \mathbf{j}_+\end{pmatrix} \;=\; \sum_{i=1}^{d+1}\gamma_i\,R_i^-$$

solves the linearized jump conditions in (3.43) *for some*

$$\dot\chi(t,y)\;=\;X\,\mathrm{e}^{\mathrm{i}(\eta_0 t\,+\,\check\eta\cdot y)}$$

such that the frequency $\eta_0\neq 0$ *and the wave vector* $\check\eta$ *satisfy* (3.45), *if and only if,*

$$c_r^2c_l^2\eta_0^2\,-\,v_rv_l\alpha_l\alpha_r\;=\;0\,, \tag{3.50}$$

where $\alpha_{l,r}$ *are defined as in* (3.46), $\gamma_i=0$ *for* $i\geq 3$, *and*

$$\gamma_1\;=\;-\,v_r\,\alpha_r\,-\,\mathrm{i}\eta_0\,c_r^2\,,\quad \gamma_2\;=\;v_l\,\alpha_l\,-\,\mathrm{i}\eta_0\,c_l^2\,. \tag{3.51}$$

Proof. This is part of the main result in [3], in different variables though. Let us give a sketch of computations. First recall that the Rankine-Hugoniot conditions (3.41) imply that

$$[p]\;=\;v_lv_r\,[\rho]\,,\qquad \left[g+\frac{1}{2}v^2\right]\;=\;0\,,$$

where $g(\rho):=f(\rho)+\frac{p(\rho)}{\rho}$ (which corresponds to the chemical potential of the fluid). These jump relations will enable us to eliminate $\dot\chi=X\,\mathrm{e}^{\mathrm{i}(\eta_0 t\,+\,\check\eta\cdot y)}$

from (3.43). Indeed, subtracting $(g + \frac{1}{2}v^2)$ times the first equation to the $(d+2)$th equation in (3.43), we may replace the latter by

$$- [p]\, \partial_t \dot\chi \;=\; \big[(c^2 - v^2) v \dot\rho \,+\, v^2\, \mathbf{j}_d\big]\;.$$

Then, substituting $X\, \mathrm{e}^{\mathrm{i}(\eta_0 t + \check\eta \cdot y)}$ for $\dot\chi$ in (3.43), we can complete the elimination of $\dot\chi$. Since $\eta_0 \neq 0$ (and therefore also $\check\eta \neq 0$ by (3.45)), we have

$$X = \frac{[\mathbf{j}_d]}{\mathrm{i}\eta_0 [\rho]}\,,$$

and we are left with the following algebraic system for $(\dot\rho, \mathbf{j})$,

$$\begin{cases} \eta_0\, [v\check{\mathbf{j}}] \cdot \check\eta \,-\, \|\check\eta\|^2\, v_l v_r\, [\mathbf{j}_d] \;=\; 0\,, \\[2ex] \big[(c^2 - v^2)\, \dot\rho \,+\, 2\, v \mathbf{j}_d\big] \;=\; 0\,, \\[2ex] \big[(c^2 - v^2) v\, \dot\rho \,+\, (v^2 + v_l v_r)\, \mathbf{j}_d\big] \;=\; 0\,, \end{cases}$$

with the additional condition that $[v\check{\mathbf{j}}]$ be colinear to $\check\eta$. The rest of the proof is a matter of elementary algebra and is left to the reader. □

Remark 3.2 *As was observed in [3], the linear surface waves found in Lemma 3.1 are slow, in that for $(\eta_0, \check\eta) \in \mathbb{R}^d$ solution of* (3.50) *with* (3.45) *and* (3.46)*, we have the inequality*

$$\eta_0^2 \;<\; v_l\, v_r\, \|\check\eta\|^2\,. \tag{3.52}$$

This inequality will be used later on.

Notice that, in terms of the abstract form (3.44) of (3.43) for $\dot\chi = \dot\chi(\eta_0 t + \check\eta \cdot y)$, the method of elimination used in Lemma 3.1 above leads to the system

$$(\partial_\xi \dot\chi)\, \widetilde{J}(\underline{u})\, \eta + \widetilde{H}(\underline{u}, \eta) \cdot (\dot\rho, \mathbf{j}) \;=\; 0_{d+2}\,,$$

with

$$\widetilde{J}(\underline{u})\, \eta \;=\; \begin{pmatrix} -\,\eta_0\, [\rho] \\ 0_{d-1} \\ 0 \\ 0 \end{pmatrix}\,,$$

$$\widetilde{H}(\underline{u}, \eta) \cdot (\dot\rho, \mathbf{j}) \;=\; \begin{pmatrix} [\mathbf{j}_d] \\ \Pi_\eta [v\check{\mathbf{j}}] \\ \eta_0\, [v\check{\mathbf{j}}] \cdot \check\eta - \|\check\eta\|^2\, v_l\, v_r\, [\mathbf{j}_d] \\ \big[(c^2 - v^2)\, \dot\rho \,+\, 2\, v \mathbf{j}_d\big] \\ \big[(c^2 - v^2) v\, \dot\rho \,+\, (v^2 + v_l v_r)\, \mathbf{j}_d\big] \end{pmatrix}\,,$$

where the brackets stand as usual for "jumps" (e.g. $[\mathbf{j}_d] = (\mathbf{j}_d)_+ - (\mathbf{j}_d)_-$), and Π_η denotes the $(d-2)\times(d-1)$ matrix whose rows are ${}^t\check{\mathrm{e}}_i$ for $i = 2, \dots, d-1$. (Recall that $(\check{\mathrm{e}}_2, \dots, \check{\mathrm{e}}_{d-1})$ has been chosen to be a basis of $\check{\eta}^\perp$.) We have $\widetilde{H}(\underline{u}, \eta) = E(\underline{u}, \eta) H(\underline{u}, \eta)$ and $\widetilde{J}(\underline{u}) = E(\underline{u}, \eta) J(\underline{u})$, with

$$E(\underline{u}, \eta) = \begin{pmatrix} 1 & 0^*_{d-1} & 0 & 0 \\ 0_{d-2} & \Pi_\eta & 0 & 0 \\ -\|\check{\eta}\|^2 v_l v_r & \eta_0\,{}^t\check{\eta} & 0 & 0 \\ 0 & 0^*_{d-1} & 1 & 0 \\ v_l v_r - (g + \frac{1}{2} v^2) & 0^*_{d-1} & 0 & 1 \end{pmatrix}.$$

Denoting by $\mathcal{C}(\underline{u}, \eta)$ the linear mapping

$$(\dot{\rho}, \mathbf{j}) \;\mapsto\; \begin{pmatrix} \Pi_\eta [v \check{\mathbf{j}}] \\ \eta_0\, [v \check{\mathbf{j}}] \cdot \check{\eta} - \;\|\check{\eta}\|^2\, v_l\, v_r\, [\mathbf{j}_d] \\ \big[(c^2 - v^2)\, \dot{\rho} \,+\, 2\, v \mathbf{j}_d\big] \\ \big[(c^2 - v^2) v\, \dot{\rho} \,+\, (v^2 + v_l v_r)\, \mathbf{j}_d\big] \end{pmatrix},$$

(that is, we retain all but the first row in $\widetilde{H}(\underline{u}, \eta) \cdot (\dot{\rho}, \mathbf{j})$), Lemma 3.1 says that the matrix made up with the column vectors $(\mathcal{C}(\underline{u}, \eta) R_1^-, \dots, \mathcal{C}(\underline{u}, \eta) R_{d+1}^-)$ is of rank d and

$$\gamma_1\, \mathcal{C}(\underline{u}, \eta) R_1^- \;+\; \gamma_2\, \mathcal{C}(\underline{u}, \eta) R_2^- \;=\; 0\,.$$

By definition (see (2.13)), the vector $\sigma = (\sigma_{-d+2}, \dots, \sigma_{-1}, \sigma_1, \sigma_2, \sigma_3)$ must be orthogonal (in $\mathbb{C}^{d+1}$ equipped with the standard hermitian product) to

$$\mathcal{C}(\underline{u}, \eta) R_1^- \;=\; \begin{pmatrix} 0_{d-2} \\ \|\check{\eta}\|^2 v_l\, (v_r\, \alpha_l - \mathrm{i}\eta_0 c_l^2) \\ -v_l \alpha_l + \mathrm{i}\eta_0 c_l^2 \\ v_l(-v_r \alpha_l + \mathrm{i}\eta_0 c_l^2) \end{pmatrix},$$

$$\mathcal{C}(\underline{u}, \eta) R_2^- \;=\; \begin{pmatrix} 0_{d-2} \\ \|\check{\eta}\|^2 v_r\, (v_l\, \alpha_r + \mathrm{i}\eta_0 c_r^2) \\ -v_r \alpha_r - \mathrm{i}\eta_0 c_r^2 \\ -v_r(v_l \alpha_r + \mathrm{i}\eta_0 c_r^2) \end{pmatrix},$$

$$\mathcal{C}(\underline{u}, \eta) R_i^- \;=\; \begin{pmatrix} \eta_0 v_r \Pi_\eta \check{\mathrm{e}}_{i-2} \\ (\eta_0^2 - v_l v_r \|\check{\eta}\|^2)\, v_r\, (\check{\eta} \cdot \check{\mathrm{e}}_{i-2}) \\ 2 v_r^2 (\check{\eta} \cdot \check{\mathrm{e}}_{i-2}) \\ v_r^2 (v_l + v_r)(\check{\eta} \cdot \check{\mathrm{e}}_{i-2}) \end{pmatrix},\ i = 3, \dots, d+1,$$

or, with the choice of the vectors $\check{\mathrm{e}}_j$ made above,

$$\mathcal{C}(\underline{u}, \eta) R_3^- \;=\; \begin{pmatrix} 0_{d-2} \\ (\eta_0^2 - v_l v_r \|\check{\eta}\|^2)\, v_r\, \|\check{\eta}\|^2 \\ 2 v_r^2\, \|\check{\eta}\|^2 \\ v_r^2 (v_l + v_r)\, \|\check{\eta}\|^2 \end{pmatrix},\ \mathcal{C}(\underline{u}, \eta) R_i^- \;=\; \begin{pmatrix} \eta_0 v_r \Pi_\eta \check{\mathrm{e}}_{i-2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for $i = 4, \ldots, d+1$. Since the $(d-2) \times (d-2)$ matrix $(\Pi_\eta \check{e}_2, \ldots, \Pi_\eta \check{e}_{d-1})$ is nonsingular, we easily see that σ must be of the form $\sigma = (0, \ldots, 0, \sigma_1, \sigma_2, \sigma_3)$. Furthermore, taking for instance

$$\sigma_2 = -v_l \alpha_r + \mathrm{i}\eta_0 c_r^2 \,,$$

we find that

$$\sigma_3 - \sigma_1 \|\check{\eta}\|^2 = \alpha_r - \mathrm{i}\eta_0 c_r^2 / v_r \; = \; -\frac{\overline{\gamma_1}}{v_r} \,,$$

and

$$\sigma_1 \; = \; -(v_r - v_l) \frac{v_r \alpha_r + \mathrm{i}\eta_0 c_r^2}{\eta_0^2 + \|\check{\eta}\|^2 v_r^2} \; = \; \gamma_1 \frac{v_r - v_l}{\eta_0^2 + \|\check{\eta}\|^2 v_r^2} \,.$$

Knowing that $\sigma^* \mathcal{C}(\underline{u}, \eta) R_{1,2}^- = 0$, that $\mathcal{C}(\underline{u}, \eta)$ has real coefficients and that $R_{1,2}^+ = \overline{R_{1,2}^-}$, we readily get

$$\begin{aligned}
\sigma^* \mathcal{C}(\underline{u}, \eta) R_1^+ \; &= \; 2\, \sigma^* \mathrm{Re} \left(\mathcal{C}(\underline{u}, \eta) R_1^- \right) \; = \; 2\, \alpha_l \, v_l \, (v_r \overline{\sigma_1} \|\check{\eta}\|^2 - \overline{\sigma_2} - v_r \overline{\sigma_3}) \\
&= \; -\, 2\, \alpha_l \, \alpha_r \, v_l \, (v_r - v_l) \,, \\
\sigma^* \mathcal{C}(\underline{u}, \eta) R_2^+ \; &= \; 2\, \sigma^* \mathrm{Re} \left(\mathcal{C}(\underline{u}, \eta) R_2^- \right) \; = \; 2\, \alpha_r \, v_r \, (v_l \overline{\sigma_1} \|\check{\eta}\|^2 - \overline{\sigma_2} - v_l \overline{\sigma_3}) \\
&= \; 2\, \mathrm{i}\, \eta_0 \, c_r^2 \, \alpha_r \, (v_r - v_l) \,.
\end{aligned}$$

Finally, since

$$\mathcal{C}(\underline{u}, \eta) R_3^+ \; = \; \begin{pmatrix} 0_{d-2} \\ -(\eta_0^2 - v_l v_r \|\check{\eta}\|^2) \, v_l \, \|\check{\eta}\|^2 \\ -2 v_l^2 \, \|\check{\eta}\|^2 \\ -v_l^2 (v_l + v_r) \, \|\check{\eta}\|^2 \end{pmatrix}$$

resembles $\mathcal{C}(\underline{u}, \eta) R_3^-$, it is not difficult to evaluate

$$\sigma^* \mathcal{C}(\underline{u}, \eta) R_3^+ \; = \; \overline{\gamma_1} \, \frac{[v]^2}{\eta_0^2 + \|\check{\eta}\|^2 v_r^2} \, \|\check{\eta}\|^2 \, \frac{v_l}{v_r} \, (v_l v_r \|\check{\eta}\|^2 - \eta_0^2) \,.$$

(Observe that by (3.52), the last factor here above is positive.)

To summarize we have

$$\left\{ \begin{aligned}
&\sigma^* \mathcal{C}(\underline{u}, \eta) R_1^+ \; = \; -\, 2\, \alpha_l \, \alpha_r \, v_l \, [v] \,, \\
&\sigma^* \mathcal{C}(\underline{u}, \eta) R_2^+ \; = \; 2\, \mathrm{i}\, \eta_0 \, c_r^2 \, \alpha_r \, [v] \,, \\
&\sigma^* \mathcal{C}(\underline{u}, \eta) R_3^+ \; = \; \overline{\gamma_1} \, \|\check{\eta}\|^2 \, \frac{v_l}{v_r} \, \frac{v_l v_r \|\check{\eta}\|^2 - \eta_0^2}{\eta_0^2 + \|\check{\eta}\|^2 v_r^2} \, [v]^2 \,, \\
&\sigma^* \mathcal{C}(\underline{u}, \eta) R_j^+ \; = \; 0 \,, \; j \in \{4, \ldots, d+1\} \,.
\end{aligned} \right. \tag{3.53}$$

3.4 Weakly nonlinear surface waves for phase boundaries

We are now going to derive the explicit form of the first and second order systems associated with our specific free boundary problem (3.42). Since the boundary condition in (3.42) is not as decoupled as in (2.8), the resulting second order system will look slightly more complicated than (2.16).

To avoid multiple indices we prefer using here notations with dots instead of subscripts 1 and 2 for the first order and second orders of the expansion. Then the first and second order systems associated with (3.42) are of the form

$$\left\{\begin{array}{lcll} \mathcal{L}(\underline{u},\eta)\cdot(\dot\rho,\dot{\mathbf{j}}) & = & 0_{2d+2} & ,\quad z>0\,, \\ \\ (\partial_\xi\dot\chi)\,J(\underline{u})\eta\,+H(\underline{u})\cdot(\dot\rho,\dot{\mathbf{j}}) & = & 0_{d+2} & ,\quad z=0\,, \end{array}\right. \tag{3.54}$$

$$\left\{\begin{array}{lcll} \mathcal{L}(\underline{u},\eta)\cdot(\ddot\rho,\ddot{\mathbf{j}})+\mathcal{M}(\underline{u},\eta;\dot\rho,\dot{\mathbf{j}},\partial_\xi\dot\chi)\cdot(\dot\rho,\dot{\mathbf{j}}) & = & 0_{2d+2} & ,\quad z>0\,, \\ \\ (\partial_\xi\ddot\chi)\,J(\underline{u})\eta\,+H(\underline{u})\cdot(\ddot\rho,\ddot{\mathbf{j}})+G(\underline{u},\eta;\dot\rho,\dot{\mathbf{j}},\partial_\tau\dot\chi,\partial_\xi\dot\chi) & = & 0_{d+2} & ,\quad z=0\,, \end{array}\right. \tag{3.55}$$

with

$$(\dot\rho,\dot{\mathbf{j}}):=\begin{pmatrix}\dot\rho_-\\ \dot{\mathbf{j}}_-\\ \dot\rho_+\\ \dot{\mathbf{j}}_+\end{pmatrix}\,,\quad (\ddot\rho,\ddot{\mathbf{j}}):=\begin{pmatrix}\ddot\rho_-\\ \ddot{\mathbf{j}}_-\\ \ddot\rho_+\\ \ddot{\mathbf{j}}_+\end{pmatrix}\,.$$

The linear terms in (3.54) and (3.55) are of course reminiscent of the linearized problem considered in Section 3.3. The operator $\mathcal{L}(\underline{u},\eta)$ is block-diagonal and defined by

$$\mathcal{L}(\underline{u},\eta)=\left(\begin{array}{c|c}\mathcal{L}_-(\rho_l,v_l,\eta) & 0\\ \hline 0 & \mathcal{L}_+(\rho_r,v_r,\eta)\end{array}\right)\,,$$

$$\mathcal{L}_\pm(\rho,v,\eta)\;=\;\left(\begin{array}{c|c|c} \eta_0\partial_\xi & \check\eta\partial_\xi & \pm\partial_z \\ \hline p'(\rho)\,{}^t\check\eta\partial_\xi & (\eta_0\partial_\xi\pm v\partial_z)\mathrm{I}_{d-1} & 0_{d-1} \\ \hline \pm(p'(\rho)-v^2)\partial_z & v\check\eta\partial_\xi & \eta_0\partial_\xi\pm 2v\partial_z \end{array}\right)\,.$$

In the boundary condition we have

$$J(\underline{u})\eta:=\begin{pmatrix}-\eta_0\,[\rho]\\ -[p]\check\eta\\ 0\\ -\eta_0\,[\rho(g+\frac12 v^2)-p]\end{pmatrix}\,,$$

$$H(\underline{u})\cdot(\ddot\rho,\ddot{\mathbf{j}})\;=\;\begin{pmatrix}[\ddot{\mathbf{j}}_d]\\ [v\,\check{\ddot{\mathbf{j}}}]\\ \big[(c^2-v^2)\,\ddot\rho\,+\,2\,v\ddot{\mathbf{j}}_d\big]\\ \big[(c^2-v^2)v\,\ddot\rho\,+\,(g+\frac32 v^2)\,\ddot{\mathbf{j}}_d\big]\end{pmatrix}\,.$$

We recall that c denotes the sound speed ($(c(\rho)^2 = p'(\rho)$) and g denotes the chemical potential ($g(\rho) = f(\rho) + \frac{p(\rho)}{\rho}$) of the fluid.

The other terms in (3.55) are of the form

$$\begin{aligned}\mathcal{M}(\underline{u},\eta;\dot\rho,\mathbf{j},\partial_\xi\dot\chi)\cdot(\dot\rho,\mathbf{j}) \;=\; & \partial_\tau(\dot\rho,\mathbf{j}) + \partial_\xi(\mathcal{Q}(\underline{u},\eta)(\dot\rho,\mathbf{j})) + \partial_z(\mathcal{P}(\underline{u})(\dot\rho,\mathbf{j})) \\ & -(\partial_\xi\dot\chi)\,\partial_z(\mathcal{N}(\underline{u},\eta)(\dot\rho,\mathbf{j}))\,,\end{aligned}$$

where $\mathcal{Q}(\underline{u},\eta)$ and $\mathcal{P}(\underline{u})$ are quadratic mappings and $\mathcal{N}(\underline{u},\eta)$ is a linear mapping, and

$$G(\underline{u},\eta;\dot\rho,\mathbf{j},\partial_\tau\dot\chi,\partial_\xi\dot\chi) \;=\; (\partial_\tau\dot\chi)\,\mathbf{e} \,-\, (\partial_\xi\dot\chi)\,N(\underline{u},\eta)(\dot\rho,\mathbf{j}) \,+\, P(\underline{u})(\dot\rho,\mathbf{j})\,,$$

where $P(\underline{u})$ is another quadratic map (involving $\mathcal{P}(\underline{u})$), $N(\underline{u},\eta)$ is another linear map (involving $\mathcal{N}(\underline{u},\eta)$), and

$$\mathbf{e}(\underline{u}) := \begin{pmatrix} -[\rho] \\ 0_d \\ -[\rho(g+\frac{1}{2}v^2)-p] \end{pmatrix}.$$

Explicit formulas are

$$\mathcal{N}(\underline{u},\eta)(\dot\rho,\mathbf{j}) = \begin{pmatrix} -\mathcal{N}_0(\rho_l,v_l,\eta)(\dot\rho,\mathbf{j}) \\ \mathcal{N}_0(\rho_r,v_r,\eta)(\dot\rho,\mathbf{j}) \end{pmatrix},$$

$$\mathcal{N}_0(\rho,v,\eta)(\dot\rho,\mathbf{j}) \;=\; \begin{pmatrix} \eta_0\dot\rho + \check\eta\cdot\check{\mathbf{j}} \\ \eta_0\check{\mathbf{j}} + p'(\rho)\dot\rho\check\eta \\ \eta_0\mathbf{j}_d + v\check\eta\cdot\check{\mathbf{j}} \end{pmatrix},$$

$$\mathcal{Q}(\underline{u},\eta)(\dot\rho,\mathbf{j}) \;=\; \begin{pmatrix} \mathcal{Q}_0(\rho_l,v_l,\eta)(\dot\rho,\mathbf{j}) \\ \mathcal{Q}_0(\rho_r,v_r,\eta)(\dot\rho,\mathbf{j}) \end{pmatrix},$$

$$\mathcal{Q}_0(\rho,v,\eta)(\dot\rho,\mathbf{j}) \;=\; \begin{pmatrix} 0 \\ \frac{1}{\rho}(\check\eta\cdot\check{\mathbf{j}})\check{\mathbf{j}} + \frac{1}{2}p''(\rho)(\dot\rho)^2\check\eta \\ \frac{1}{\rho}(\check\eta\cdot\check{\mathbf{j}})(\mathbf{j}_d - v\dot\rho) \end{pmatrix},$$

$$\mathcal{P}(\underline{u})(\dot\rho,\mathbf{j}) = \begin{pmatrix} -\mathcal{P}_0(\rho_l,v_l)(\dot\rho,\mathbf{j}) \\ \mathcal{P}_0(\rho_r,v_r)(\dot\rho,\mathbf{j}) \end{pmatrix},$$

$$\mathcal{P}_0(\rho,v)(\dot\rho,\mathbf{j}) \;=\; \begin{pmatrix} 0 \\ \frac{1}{\rho}(\mathbf{j}_d - v\dot\rho)\check{\mathbf{j}} \\ (\frac{1}{2}p''(\rho) + \frac{v^2}{\rho})(\dot\rho)^2 + \frac{1}{\rho}\mathbf{j}_d(\mathbf{j}_d - 2v\dot\rho) \end{pmatrix},$$

$$P(\underline{u})(\dot\rho,\mathbf{j}) \;=\; \begin{pmatrix} \mathcal{P}_0(\rho_r,v_r)(\dot\rho,\mathbf{j}) - \mathcal{P}_0(\rho_l,v_l)(\dot\rho,\mathbf{j}) \\ \pi(\rho_r,v_r)(\dot\rho,\mathbf{j}) - \pi(\rho_l,v_l)(\dot\rho,\mathbf{j}) \end{pmatrix},$$

$$\pi(\rho,v)(\dot\rho,\mathbf{j}) \;=\; \frac{v}{\rho}(\frac{1}{2}\|\mathbf{j}\|^2 + (\mathbf{j}_d)^2) + \frac{1}{2}v(p''(\rho) - \frac{p'(\rho)-3v^2}{\rho})(\dot\rho)^2 + \frac{p'(\rho)-3v^2}{\rho}\,\dot\rho\,\mathbf{j}_d\,,$$

$$N(\underline{u},\eta)(\dot\rho,\mathbf{j}) \;=\; \begin{pmatrix} \mathcal{N}_0(\rho_r,v_r,\eta)(\dot\rho,\mathbf{j}) - \mathcal{N}_0(\rho_l,v_l,\eta)(\dot\rho,\mathbf{j}) \\ \nu(\rho_r,v_r,\eta)(\dot\rho,\mathbf{j}) - \nu(\rho_l,v_l,\eta)(\dot\rho,\mathbf{j}) \end{pmatrix},$$

$$\nu(\rho,v,\eta)(\dot\rho,\mathbf{j}) \;=\; \eta_0(g(\rho) - \tfrac{1}{2}v^2)\,\dot\rho \;+\; (g(\rho)+\tfrac{1}{2}v^2)\,(\check\eta\cdot\check{\mathbf{j}}) \;+\; \eta_0\, v\, \mathbf{j}_d\,.$$

From Lemma 3.1 the resolution of (3.54) is given by the following analogue of Proposition 2.1.

Proposition 3.3 *The solutions $(\xi,z)\mapsto(\dot\rho,\mathbf{j},\dot\chi)(\xi,z)$ of (3.54) that are square integrable in ξ and such that $\dot\rho$, $\mathbf{j}$ go to zero as $z\to+\infty$ are of the form*

$$(\dot\rho,\mathbf{j})(\xi,z) \;=\; (w *_\xi r)(\xi,z)\,, \quad \dot\chi(\xi) \;=\; (w *_\xi s)(\xi)\,,$$

$$\widehat{r}(k,z) \;=\; \begin{cases} \gamma_1\, \mathrm{e}^{\beta_1^-\,k\,z}\, R_1^- \;+\; \gamma_2\, \mathrm{e}^{\beta_2^-\,k\,z}\, R_2^-\,, & k\;>\;0\,,\, z>0\,, \\ \overline{\gamma_1}\, \mathrm{e}^{\beta_1^+\,k\,z}\, R_1^+ \;+\; \overline{\gamma_2}\, \mathrm{e}^{\beta_2^+\,k\,z}\, R_2^+\,, & k\;<\;0\,, z>0\,, \end{cases}$$

$$\widehat{s}(k) \;=\; \begin{cases} -\,\dfrac{\gamma_2\alpha_r+\gamma_1\alpha_l}{\mathrm{i}k\eta_0[\rho]}\,, & k\;>\;0\,, \\[2ex] -\,\dfrac{\overline{\gamma_2}\alpha_r+\overline{\gamma_1}\alpha_l}{\mathrm{i}k\eta_0[\rho]}\,, & k\;<\;0\,, \end{cases}$$

where w is an arbitrary L^2 function.

Now, since the triangular matrix $E(\underline{u},\eta)$ is nonsingular (for $\eta\neq 0$), the boundary condition in the second order system (3.55) is equivalent to

$$(\partial_\xi\ddot\chi)\,\widetilde{J}(\underline{u})\eta + \widetilde{H}(\underline{u},\eta)\cdot(\ddot\rho,\ddot{\mathbf{j}}) + \widetilde{G}(\underline{u},\eta;\dot\rho,\mathbf{j},\partial_\tau\dot\chi,\partial_\xi\dot\chi) = 0_{d+2}\,,$$

with $\widetilde{G}(\underline{u},\eta;\cdot) = E(\underline{u},\eta)G(\underline{u},\eta;\cdot)$ (and as before $\widetilde{J}(\underline{u}) = E(\underline{u},\eta)J(\underline{u})$, $\widetilde{H}(\underline{u},\eta) = E(\underline{u},\eta)H(\underline{u},\eta)$), or, isolating the first row,

$$\begin{cases} -\eta_0[\rho](\partial_\xi\ddot\chi) + [\ddot{\mathbf{j}}_d] - \eta_0[\rho](\partial_\tau\dot\chi) \;-\; [\eta_0\dot\rho + \check\eta\cdot\check{\mathbf{j}}]\,(\partial_\xi\dot\chi) \;=\; 0\,, \\[2ex] \mathcal{C}(\underline{u},\eta)\cdot(\ddot\rho,\ddot{\mathbf{j}}) + \widetilde{g}(\underline{u},\eta;\dot\rho,\mathbf{j},\partial_\tau\dot\chi,\partial_\xi\dot\chi) = 0_{d+1}\,, \end{cases} \tag{3.56}$$

where

$$\widetilde{g}(\underline{u},\eta;\dot\rho,\mathbf{j},\partial_\tau\dot\chi,\partial_\xi\dot\chi) \;:=\; e(\underline{u},\eta)\,G(\underline{u},\eta;\dot\rho,\mathbf{j},\partial_\tau\dot\chi,\partial_\xi\dot\chi)\,.$$

$$e(\underline{u},\eta) \;:=\; \begin{pmatrix} 0_{d-2} & \Pi_\eta & 0 & 0 \\ -\|\check\eta\|^2 v_l v_r & \eta_0\,{}^t\check\eta & 0 & 0 \\ 0 & 0_{d-1}^* & 1 & 0 \\ v_l v_r - (g+\frac{1}{2}v^2) & 0_{d-1}^* & 0 & 1 \end{pmatrix}.$$

It turns out that the factor of $\partial_\tau\dot\chi$ in $\widetilde{g}$ reduces to

$$e(\underline{u},\eta)\,\mathbf{e}(\underline{u}) \;=\; \begin{pmatrix} 0_{d-2} \\ \|\check\eta\|^2 v_l v_r[\rho] \\ 0 \\ 0 \end{pmatrix}.$$

Theorem 3.4 *Under the Assumptions of Lemma 3.1, weakly nonlinear surface waves for the nonlinear model* (3.37) (3.39) *are governed by a nonlocal amplitude equation of the form* (2.30)*, where Q is related by* (2.31) *to $K := 2\pi\mathcal{F}^{-1}(\Lambda) \in \mathscr{S}'(\mathbb{R}^2)$, in which Λ is defined as in* (2.33) *by*

$$\Lambda(k,\ell) \;:=\; \frac{a_1^s(k,\ell)}{\mathrm{i}\;(k+\ell)\,a_0(k+\ell)}\,,\qquad a_0(k) = \begin{cases} \frac{\alpha_0}{k}\,, & k>0\,,\\[1ex] \frac{\overline{\alpha_0}}{k}\,, & k<0\,,\end{cases}$$

with α_0 and a_1^s defined in (3.59) (3.60) (3.61) (3.62) *below. This kernel Λ is well defined because $\alpha_0 \neq 0$. In addition, it satisfies the reality-symmetry-homogeneity properties in* (2.35)*, and the stability condition* (2.36) *is equivalent to requiring that $a(\mathrm{Re}(\gamma_1 R_1^- + \gamma_2 R_2^-))$ and $a(\mathrm{Im}(\gamma_1 R_1^- + \gamma_2 R_2^-))$ be real, with the linear form defined by* (3.63) *below.*

Proof. Similarly as in the abstract framework of Section 2.3, using the reformulation (3.56) of the boundary condition in (3.55), we find that for (3.55) to have a L^2 solution, the first order solution $(\dot\rho, \mathbf{j}, \dot\chi)$ of (3.54) must satisfy

$$\int_0^{+\infty} L(k,z)\,m_1(k,z,\tau)\,\mathrm{d}z \;+\; \sigma(k)\,g_1(k,\tau) \;=\; 0\,, \tag{3.57}$$

with

$$m_1 := \mathcal{F}\left[\mathcal{M}(\underline{u},\eta;\dot\rho,\mathbf{j},\partial_\xi\dot\chi)\cdot(\dot\rho,\mathbf{j})\right]\,,\quad g_1 := \mathcal{F}[\widetilde{g}(\underline{u},\eta;\dot\rho,\mathbf{j},\partial_\tau\dot\chi,\partial_\xi\dot\chi)]\,,$$

$$L(k,z) \;:=\; \sum_{j=1}^{3} \frac{\sigma^*\mathcal{C}(\underline{u},\eta)\,R_j^+}{(L_j^+)^*\breve{\mathbb{A}}^d R_j^+}\,\mathrm{e}^{-k\beta_j^+ z}\,(L_j^+)^*\,,\quad k>0\,,$$

$$L(k,z) \;:=\; \overline{L(-k,z)}\,,\quad k<0\,,$$

$$\sigma(k) \;:=\; \sigma^*\,,\ k>0\,,\ \text{ and } \sigma(k) \;:=\; \overline{\sigma(k)}\,,\ k<0\,.$$

(The fact that the sum is limited to $j \leq 3$ comes from $\sigma^*\mathcal{C}(\underline{u},\eta)R_j^+ = 0$ for $j \geq 4$, see (3.53).)

Substituting $\widehat{w}(k,\tau)\,\widehat{r}(k,z)$ for $\mathcal{F}[(\dot\rho,\mathbf{j})](k,z,\tau)$, and $\widehat{w}(k,\tau)\ \widehat{s}(k)$ for $\mathcal{F}[\dot\chi](k)$ in the definition of m_1 and g_1, we can rewrite (3.57) as

$$a_0(k)\,\partial_\tau\widehat{w}(k,\tau) \;+\; \int_{-\infty}^{+\infty} a_1^s(k-\ell,\ell)\ \widehat{w}(k-\ell,\tau)\widehat{w}(\ell,\tau)\,\mathrm{d}\ell \;=\; 0\,,$$

with

$$a_0(k) \;:=\; \int_0^{+\infty} L(k,z)\,\widehat{r}(k,z)\,\mathrm{d}z \;+\; \widehat{s}(k)\,\sigma(k)\,e(\underline{u},\eta)\mathbf{e}(\underline{u})\,, \tag{3.58}$$

$$a_1^s(k,\ell) \;:=\; \int_0^{+\infty} L(k+\ell,z)\,m^s(\underline{u},\eta;k,\ell,z)\,\mathrm{d}z \;+\; \sigma(k+\ell)\,e(\underline{u},\eta)\,\gamma(\underline{u},\eta;k,\ell)\,, \tag{3.59}$$

$$
\begin{aligned}
2\, m^s(\underline{u},\eta;k,\ell,z) \;&:=\; \mathrm{i}(k+\ell)\,\mathcal{Q}_2(\underline{u},\eta)(\widehat{r}(k,z),\widehat{r}(\ell,z)) \qquad (3.60)\\
&+\mathrm{i}\ell\,\mathcal{P}_2(\underline{u})(\widehat{r}(k,z),\mathcal{F}[r'](\ell,z)) \;+\; \mathrm{i}k\,\mathcal{P}_2(\underline{u})(\widehat{r}(\ell,z),\mathcal{F}[r'](k,z))\\
&-\mathrm{i}\ell\,\mathrm{i}k\widehat{s}(k)\,\mathcal{N}(\underline{u},\eta)(\mathcal{F}[r'](\ell,z)) \;-\; \mathrm{i}k\,\mathrm{i}\ell\widehat{s}(\ell)\,\mathcal{N}(\underline{u},\eta)(\mathcal{F}[r'](k,z))\,,
\end{aligned}
$$

$$
\begin{aligned}
2\gamma(\underline{u},\eta;k,\ell) \;&:=\; 2P_2(\underline{u})(\widehat{r}(k,0),\widehat{r}(\ell,0)) \qquad (3.61)\\
&-\mathrm{i}k\widehat{s}(k)\;N(\underline{u},\eta)(\widehat{r}(\ell,0)) \;-\; \mathrm{i}\ell\widehat{s}(\ell)\;N(\underline{u},\eta)(\widehat{r}(k,0))\,,
\end{aligned}
$$

where $\mathcal{Q}_2(\underline{u},\eta)$, $\mathcal{P}_2(\underline{u})$, and $P_2(\underline{u})$ denote the symmetric bilinear mappings associated with the quadratic mappings $\mathcal{Q}(\underline{u},\eta)$, $\mathcal{P}(\underline{u})$, and $P(\underline{u})$ respectively, and

$$
\mathrm{i}\;\ell\;\mathcal{F}[r'](\ell,z) \;:=\; \partial_z\widehat{r}(\ell,z)
$$

(unlike what we did in the abstract framework of Section 2.3 we do not insert the matrix $\breve{\mathbb{I}}_{2n}$ here). By direct computation we find that $a_0(k)=\alpha_0/k$ for $k>0$, and $a_0(k)=\overline{\alpha_0}/k$ for $k>0$, with

$$
\begin{aligned}
\alpha_0 \;:=\; &\frac{\sigma^*\mathcal{C}(\underline{u},\eta)\,R_1^+}{\beta_1^+-\beta_1^-}\,\frac{\gamma_1(L_1^+)^*R_1^-}{(L_1^+)^*\breve{\mathbb{A}}^dR_1^+} \;+\; \frac{\sigma^*\mathcal{C}(\underline{u},\eta)\,R_2^+}{\beta_2^+-\beta_2^-}\,\frac{\gamma_2(L_2^+)^*R_2^-}{(L_2^+)^*\breve{\mathbb{A}}^dR_2^+} \qquad (3.62)\\
&+\frac{\sigma^*\mathcal{C}(\underline{u},\eta)\,R_3^+}{\beta_3^+-\beta_1^-}\,\frac{\gamma_1(L_3^+)^*R_1^-}{(L_3^+)^*\breve{\mathbb{A}}^dR_3^+} \;+\; \widehat{s}(1)\,\sigma^*e(\underline{u},\eta)\mathbf{e}(\underline{u})\,.
\end{aligned}
$$

(We have used here the observation that $(L_2^+)^*R_1^-=0$ and $(L_j^+)^*R_2^-=0$ for $j=1$ or $j\geq 3$, which is an obvious consequence of the 'block form' of these vectors.) To check whether the number α_0 is nonzero we recall from (3.46), (3.49), (3.51), and (3.53) the values of $\beta_j^\pm$, γ_p, $(L_j^+)^*\breve{\mathbb{A}}^dR_j^+$ and $\sigma^*\mathcal{C}(\underline{u},\eta)$ respectively. In particular, we observe that

$$
\beta_1^+-\beta_1^- \;=\; \frac{2\alpha_l}{c_l^2-v_l^2}\,,\quad \beta_2^+-\beta_2^- \;=\; \frac{2\alpha_r}{c_r^2-v_r^2}\,,
$$

and thus

$$
(L_1^+)^*\breve{\mathbb{A}}^dR_1^+ \;=\; \gamma_2\,(\beta_1^+-\beta_1^-)\,,\quad (L_2^+)^*\breve{\mathbb{A}}^dR_2^+ \;=\; -\gamma_1\,(\beta_2^+-\beta_2^-)\,.
$$

In addition, going back to the definitions (3.47) and (3.48), we easily compute that

$$
(L_1^+)^*R_1^- \;=\; \frac{2c_l^4\|\check{\eta}\|^2-v_l^2\eta_0^2}{c_l^2-v_l^2}\,,
$$

which is a positive real number by (3.52) (and $c_l^2>v_l^2$), and similarly,

$$
(L_2^+)^*R_2^- \;=\; -\,\frac{2c_r^4\|\check{\eta}\|^2-v_r^2\eta_0^2}{c_r^2-v_r^2}
$$

is a negative real number (because of (3.52) and $c_r^2 > v_r^2$). Therefore, we find that the first two terms in (3.62) equal to

$$2\,[v]\,\alpha_r\,\left(-\frac{\gamma_1}{\gamma_2}\frac{\theta_l\alpha_l v_l}{(\beta_1^+-\beta_1^-)^2}\;+\;\frac{\gamma_2}{\gamma_1}\frac{\mathrm{i}\eta_0\,\theta_r c_r^2}{(\beta_2^+-\beta_2^-)^2}\right)\,,$$

where

$$\theta_{l,r}\;:=\;\frac{2c_{l,r}^4\|\check\eta\|^2-v_{l,r}^2\eta_0^2}{c_{l,r}^2-v_{l,r}^2}\,.$$

Concerning the last term in (3.62) we find the value

$$\frac{\mathrm{i}}{\eta_0[\rho]}\,(\gamma_2\alpha_r+\gamma_1\alpha_l)\,\overline{\sigma}_1\,v_lv_r[\rho]\|\check\eta\|^2\;=\;\frac{\mathrm{i}\,[v]v_lv_r\|\check\eta\|^2}{\eta_0\,(\eta_0^2+\|\check\eta\|^2v_r^2)}\,(\alpha_l|\gamma_1|^2+\alpha_r\gamma_2\overline{\gamma_1})\,.$$

And finally, after computing that

$$(L_3^+)^*R_1^-\;=\;\frac{c_l^2\|\check\eta\|^2}{c_l^2-v_l^2}\,\overline{\gamma_2}\;\text{ and }\;\beta_3^+-\beta_1^-\;=\;\frac{\overline{\gamma_2}}{v_l(c_l^2-v_l^2)}\,,$$

we find that the third term in (3.62) equals to

$$-\,[v]^2\,|\gamma_1|^2\,c_l^2\,\|\check\eta\|^2\,\frac{v_l\,(v_lv_r\|\check\eta\|^2-\eta_0^2)}{v_r\,\mu_l\,\mu_r}\,,$$

with $\mu_{l,r}\;:=\;\eta_0^2+\|\check\eta\|^2v_{l,r}^2$. (It can easily be checked that each of these terms has the dimension of $c^8\beta^2$, or equivalently x^6t^{-8} in the physical space-time variables.) Observing that, thanks to the dispersion relation (3.50),

$$\mathrm{i}\eta_0\,\gamma_2\overline{\gamma_1}\;=\;-\,\eta_0^2\,(\alpha_lv_lc_r^2+\alpha_rv_rc_l^2)\in(-\infty,0)\,,$$

we readily find that

$$\begin{aligned}\mathrm{Re}(\alpha_0)\;=\;&-[v]\alpha_r\,\eta_0^2\,(\alpha_lv_lc_r^2+\alpha_rv_rc_l^2)\,\left(\tfrac{2\theta_rc_r^2}{|\gamma_1|^2(\beta_2^+-\beta_2^-)^2}+\tfrac{v_lv_r\|\check\eta\|^2}{\mu_r\eta_0^2}\right)\\&-[v]^2\,|\gamma_1|^2\,c_l^2\,\|\check\eta\|^2\,\tfrac{v_l\,(v_lv_r\|\check\eta\|^2-\eta_0^2)}{v_r\,\mu_l\,\mu_r}\,,\end{aligned}$$

with $\mu_{l,r}\;:=\;\eta_0^2+\|\check\eta\|^2v_{l,r}^2$,

$$\mathrm{Im}(\alpha_0)\;=\;2\,[v]\,\frac{\alpha_l}{\eta_0}\,\left(\frac{\eta_0^2\,\alpha_r}{|\gamma_2|^2}\,(\alpha_lv_lc_r^2+\alpha_rv_rc_l^2)\,\frac{\theta_l\alpha_lv_l}{(\beta_1^+-\beta_1^-)^2}\;+\;\frac{v_lv_r\|\check\eta\|^2}{\mu_r}\,|\gamma_1|^2\right)\,.$$

Since $[v]$, $\alpha_{l,r}$, $\theta_{l,r}$, and $\mu_{l,r}$ are all positive real numbers, we see that $\mathrm{Im}(\alpha_0)$ is nonzero (it is of the same sign as $[v]$). Concerning $\mathrm{Re}(\alpha_0)$, it is always nonzero for $[v]>0$, which corresponds to an expansive phase transition (typically, vaporization), and it is also nonzero for $-[v]>0$ and not too big. To evaluate

$$a_1^s(1,0\pm) \;=\; \lim_{\ell\to 0\pm} \int_0^{+\infty} L(1+\ell,z)\, m^s(\underline{u},\eta;1,\ell,z)\,\mathrm{d}z + \sigma^*\, e(\underline{u},\eta)\,\gamma(\underline{u},\eta;1,0\pm)\,,$$

we go back to the definitions (3.60) (3.61) of m^s and γ, and also to the definition of $\widehat{s}$ (see Proposition 3.3). We thus see that

$$\lim_{\ell\to 0+} \int_0^{+\infty} L(1+\ell,z)\, m^s(\underline{u},\eta;1,\ell,z)\,\mathrm{d}z \;=\; \frac{\sigma^*\underline{\mathcal{C}}R_j^+}{2(L_j^+)^*\breve{\mathbb{A}}^d R_j^+}\times$$

$$\frac{(L_j^+)^*}{\beta_j^+-\beta_p^-}\left(\mathrm{i}\underline{\mathcal{Q}_2}(V,\gamma_p R_p^-)+\underline{\mathcal{P}_2}(V,\gamma_p\beta_p^- R_p^-)+\frac{\gamma_2\alpha_r+\gamma_1\alpha_l}{\eta_0[\rho]}\,\underline{\mathcal{N}}(\gamma_p\beta_p^- R_p^-)\right)\,,$$

$$\lim_{\ell\to 0-} \int_0^{+\infty} L(1+\ell,z)\, m^s(\underline{u},\eta;1,\ell,z)\,\mathrm{d}z \;=\; \frac{\sigma^*\underline{\mathcal{C}}R_j^+}{2(L_j^+)^*\breve{\mathbb{A}}^d R_j^+}\times$$

$$\frac{(L_j^+)^*}{\beta_j^+-\beta_p^-}\left(\mathrm{i}\underline{\mathcal{Q}_2}(\overline{V},\gamma_p R_p^-)+\underline{\mathcal{P}_2}(\overline{V},\gamma_p\beta_p^- R_p^-)+\frac{\overline{\gamma_2}\alpha_r+\overline{\gamma_1}\alpha_l}{\eta_0[\rho]}\,\underline{\mathcal{N}}(\gamma_p\beta_p^- R_p^-)\right),$$

with, as before,

$$V=\gamma_1 R_1^- + \gamma_2 R_2^-\,,$$

and

$$\gamma(\underline{u},\eta;1,0+) \;=\; \underline{P_2}(V,V)+\frac{\gamma_2\alpha_r+\gamma_1\alpha_l}{\eta_0[\rho]}\,\underline{N}(V)\,,$$

$$\gamma(\underline{u},\eta;1,0-) \;=\; \underline{P_2}(V,\overline{V})+\frac{\gamma_2\alpha_r+\gamma_1\alpha_l}{2\eta_0[\rho]}\,\underline{N}(\overline{V}) \;+\; \frac{\overline{\gamma_2}\alpha_r+\overline{\gamma_1}\alpha_l}{2\eta_0[\rho]}\,\underline{N}(V)\,,$$

We may observe that $\gamma_2\alpha_r+\gamma_1\alpha_l = SV$, where $S=(-S_0,S_0)$ with $S_0=(0,0_{d-1}^*,-1)$. Therefore, Hunter's stability condition (2.36) for phase boundaries is equivalent to $a(\mathrm{Re}(V))$ and $a(\mathrm{Im}(V))$ being real, with

$$a(R) \;:=\; \sum_{j=1}^{3}\sum_{p=1}^{2}\frac{\sigma^*\underline{\mathcal{C}}R_j^+}{2\alpha_0(L_j^+)^*\breve{\mathbb{A}}^d R_j^+}\times \tag{3.63}$$

$$\frac{(L_j^+)^*}{\beta_j^+-\beta_p^-}\left((\underline{\mathcal{Q}_2}-\mathrm{i}\beta_p^-\underline{\mathcal{P}_2})(R,\gamma_p R_p^-)-\mathrm{i}\beta_p^-\frac{SR}{\eta_0[\rho]}\,\underline{\mathcal{N}}(\gamma_p R_p^-)\right)$$

$$-\frac{\mathrm{i}}{\alpha_0}\,\sigma^*\underline{e}\left(\underline{P_2}(V,R)\;+\;\frac{SV}{2\eta_0[\rho]}\,\underline{N}(R)\;+\;\frac{SR}{2\eta_0[\rho]}\,\underline{N}(V)\right).$$

$\square$

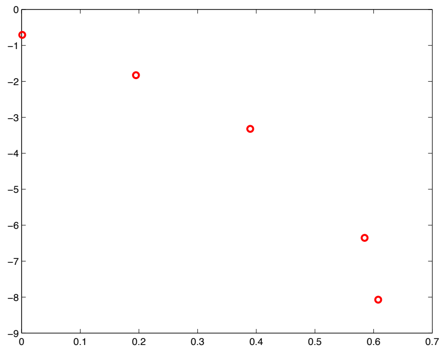

Figure 1: Ratio $\mathrm{Im}(a(\mathrm{Re}(V)))/\mathrm{Re}(a(\mathrm{Re}(V)))$ in terms of the mass transfer flux j for phase transitions in water at $T = 600K$ (thermodynamic coefficients taken from [18]).

The condition (3.63) can be tested numerically. We present on Figure 3.4 numerical results in a realistic situation, which show that (3.63) is not satisfied. This might explain why surface waves are hardly ever observed in liquid-vapor flows.

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